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Flows, Funds, and Sectorial Interdependence in the Theory of Production*

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1. INTRODUCTION AND PRINCIPAL ASSUMPTIONS

1.1. In input-output models, both theoretical and applied, assumptions about the interdependences among the various production processes are, in many cases, the only hypotheses concerning production. In particular, little or no attention is given to assumptions regarding the time profiles of the processes and their possible influence over the quantity system and the price system. Actually, in theoretical models with an input-output framework, explicit simplifying hypotheses about the time profiles of inputs and outputs are made; but this is not the case in most applications based on input-output tables, although it is recognized that the observed flows on which input coefficients are (assumed to be) obtained are the result of the arrangement of processes which do not have simple time profiles. On the other hand, the theoretical models with a linear production structure, even if not designed for application, derive much of their appeal from the fact that their structure may be thought of as derived from observations.[1] It is the aim of this paper to investigate the relations between sectorial interdependence and the time profiles of production processes. In this attempt, no space will be given to the problem of aggregation over commodities of different types, although we are aware that the two problems are connected.

1.2. More precisely, this paper tries to give answers to questions of the following kind:

— in what sense, and for what results is the information given by the

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[1] For the derivation of theoretical models from (observed) input-output tables, one may refer to L. L. Pasinetti, *Lectures on the Theory of Production*, London, Macmillan, 1977.

structure of interdependences sufficient, without any reference to time profiles?

— is there a specific time profile of inputs which is implicit in a given structure of interdependence, or may any time profile be consistent with it?

— are we allowed to assume that production, as it appears in the input-output framework, is instantaneous?

In order to try to make some of these points clearer, we shall make use of a rather general model, which takes the analytical representation of production processes from Georgescu-Roegen's flow-fund model, and the structure of growth model from neo-Austrian models of the non-integrated type. Of course, this model is not designed only for the kind of problems which will be considered here, but it offers a good basis for them also.

1.3. Let us start with some principal assumptions.

a) A (finite) list of goods is given. Goods will be considered as different if they have different physical characteristics (or, possibly, location); not for the date of their availability. It must be stressed that the list of goods is assumed to be independent of the time profile of the production processes. This assumption is particularly relevant, since one of the main objectives of this paper concerns the use of observed input-output tables, for which the definition of goods must be taken as given *a priori*, mainly on the basis of the physical characteristics of goods (the difference in location may be considered as present, in some sense, in input-output tables through the separate evaluation of the services of transportation and commerce).

b) Some, of the goods can be produced; indeed we shall assume that all goods but one (homogeneous labour) are produced, but the results can be extended to the case of many primary goods.

c) Production requires time.

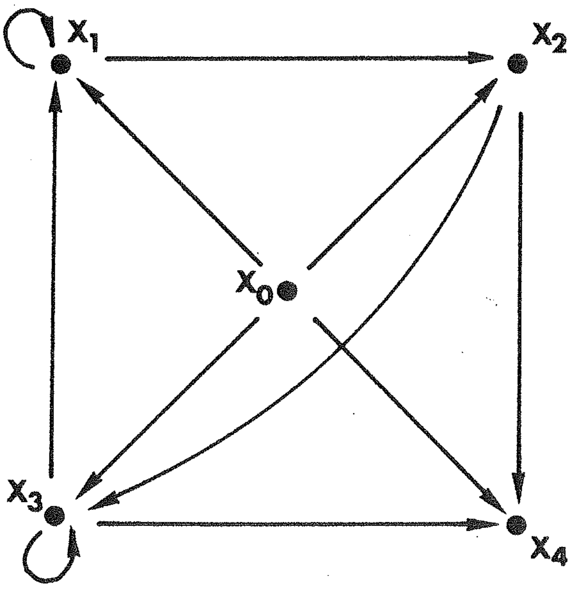
d) Production processes are single product processes, but in a broad sense; i.e. a process will be said to have a single product if the production of only one of the goods is the aim of that process. To put it more clearly: processes with fixed capital goods and processes with repeated outputs of the same type will be allowed within this assumption, provided that suitable hypotheses are introduced. In most of the paper reference will be made mainly to the case of no fixed capital; the results will be extended to the fixed capital case in the footnotes. It will also be assumed that only one technique is available for each good.[2]

[2] Since we shall assume constant returns, no joint production and one primary input, the assumption of no choice of techniques is not really restrictive, due to non substitution theorems.

e) The economic system is a closed one, i.e. no imports (or exports) are allowed (or necessary).

2. THE GRAPH OF SECTORIAL INTERDEPENDENCES

2.1. For each good, excluding labour, one production process is given, which makes use of different inputs, both produced goods and labour. When, for each producible good, we give (only) the *list* of goods used as inputs in its production, we define what may already be called, at a first level, the "sectorial interdependences structure" of the production system. The mathematical structure of this concept is completely described by a *graph* (X, Γ), where X is the set of goods and Γ is the input-output relation; i.e., for each ordered pair of goods $(x_i, x_j) \varepsilon X^2$, $(x_i, x_j) \varepsilon \Gamma$ if and only if x_i is an input in the production of x_j (fig. 1) (in the figure, x_0 is labour, and we see that it enters directly in the production of all other goods: this hypothesis could be relaxed in the usual way, by assuming that labour enters directly *or* indirectly into each production).



	x_1	x_2	x_3	x_4
x_1	1	1	0	0
x_2	0	0	1	1
x_3	1	0	1	1
x_4	0	0	0	0
x_0	1	1	1	1

Matrix of the graph

Fig. 1

2.2. Many important properties of the production system — and of the models based on it —, both on the quantity side and on the price side, may be derived from the characteristics of this graph, such as:[3]

a) identification of Sraffa's basic commodities;

[3] For an analysis of the use of graphs in input-output models, see P. TANI, *Analisi microeconomica della produzione*, Roma, La Nuova Italia Scientifica, 1986, chapter VIII.

b) indecomposability of the system and related properties (the strong connectedness of the graph is a necessary and sufficient condition for the matrix of the system to be indecomposable);

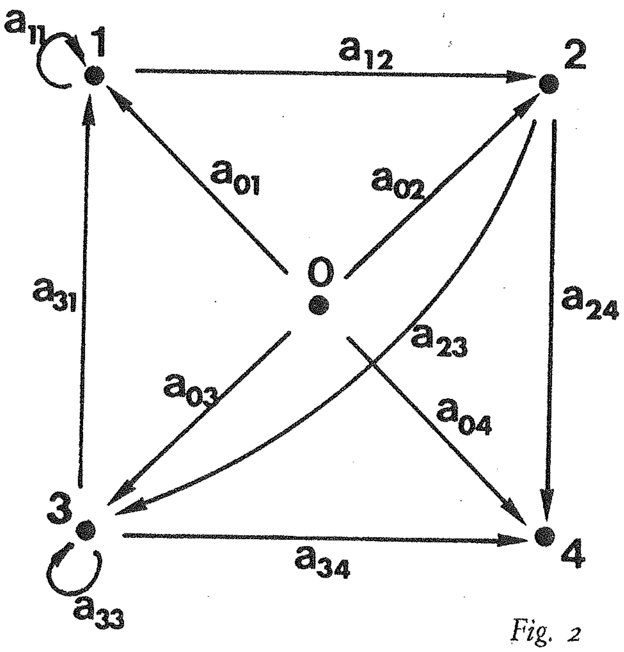
c) existence of a maximum for the system's rate of growth and rate of profit (the existence of at least one circuit in the graph is a necessary and sufficient condition);

d) possibility of defining vertical integrated processes of finite duration (the non-existence of circuits is here a necessary and sufficient condition).

All these characteristics of the production system are invariant with respect to the splitting of a process, with the introduction of the corresponding intermediate product into the list of goods.

Therefore, this first level of the concept of sectorial interdependence is a very important one, although it requires little information about the processes; in particular, it requires no information about the amount of inputs and outputs and no information about their distribution over time.

2.3. A second, richer concept of sectorial interdependence, indeed the most commonly used one, is that which corresponds to the input-output matrices in the ordinary sense (apart from problems of aggregation). Here again we may make use of the concept of a graph, and represent the sectorial interdependence by a graph to each arc of which is attached a *valuation* (i.e. a real non-negative number). In particular, the valuation of the arc (x_i, x_j) may be the coefficient a_{ij} (input coefficient of the good x_i in the production of x_j). We may assume constant returns, or a_{ij} may be the ratio of the total amount of input x_i actually used in the production of output x_j (fig. 2).



$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$a = \begin{bmatrix} a_{01} & a_{02} & a_{03} & a_{04} \end{bmatrix}$$

$$(a_{ij} = 0 \quad \text{if} \quad (x_i, x_j) \notin \Gamma)$$

Fig. 2

Here we have taken into consideration not only the existence of input x_i in the production of x_j, but also the relevant amount of this input. Again, we say nothing about the distribution of inputs and outputs over time. When information about the time profile of inputs is to be introduced, one should attach to each arc of the graph not simply a real non-negative number, but a function of time, as we shall see later on.

3. A FLOW-FUND MODEL OF PRODUCTION

3.1 More precise assumptions about the analytical representation of the production processes are now needed, and we shall refer to the flow-fund model introduced by Georgescu-Roegen.[4] According to this model, any production process is defined by giving its *border* (i.e. a specification of which operations are considered to belong to the process and which are not) and its *duration* in time. Within this duration (a finite interval $[0, T]$), anything which passes through the border — i.e. anything which goes into or comes out of the process — is recorded, so that, for each instant $t \epsilon [0, T]$, it will be possible to find out how much of each element has gone in and how much has come out since the process started. The chief distinction which is made between the elements of the process concerns the fact that, for some of them, it is possible (and indeed convenient) to assume that the total amount of input over the entire duration of the process is equal to the total amount of output. It is then possible to treat these elements as *funds*, the other elements being treated as *flows*. Georgescu-Roegen chooses to treat workers, land and the elements which represent fixed capital as funds, while raw materials, intermediate goods, energy — and, on the output side, the product itself — are treated as flows. It is clear that, as far as fixed capital is concerned, this representation requires special assumptions: within the process, operations of repair and renewal are performed, so that depreciation is always exactly compensated.[5]

[4] See, e.g., N. GEORGESCU-ROEGEN, "The Economics of Production", *American Economic Review*, May 1970; *The Entropy Law and the Economic Process*, Cambridge (Mass.), Harvard University Press, 1971, chapter IX; "Process Analysis and the Neoclassical Theory of Production", *American Journal of Agricultural Economics*, May 1972.

[5] Although the problem of fixed capital is not central in this paper, some brief remarks about the costs and benefits of an analytical representation of fixed capital by means of funds with respect to a representation of all the elements of the production process by means of flows, may be useful.

The chief drawback is connected with the assumptions, stated in the text, which are necessary in order that we may assume that the elements of fixed capital do not change their productivity over time. Utilization of used capital goods, their possible exchange, optimal truncation, are all problems whose treatment is made more difficult when the elements of fixed capital are treated as funds. On the other hand, in this case, there is no need for fictitious joint production and, above all, the characteristic of the way in which the elements of fixed capital (and, more generally, of fund elements) participate in the production process are better represented: it seems difficult

3.2. According to the assumptions that have been introduced, each process will be represented by a vector of functions of time:

$$[Q(t), \boldsymbol{F}(t), \boldsymbol{K}(t), L(t)], \quad t\varepsilon[0, T],$$

where $Q(t)$ refers to the flow of output of the product of the process; $\boldsymbol{F}(t)$ is itself a vector of functions $F_i(t)$, $i = 1, \ldots, n$ (where n is the number of producible goods), regarding the flow-inputs; $\boldsymbol{K}(t)$ is a vector of functions $K_i(t)$, $i = 1, \ldots, n$, regarding the funds of fixed capital goods; $L(t)$ refers to the workers (no land is assumed to be used).

The functions referring to the flows ($Q(t)$ and the $F_i(t)$'s) are non-decreasing functions of t, which measure, for each $t\varepsilon[0, T]$, the amount of cumulated input or output from 0 to t. Fig. 3 illustrates a possible shape of these functions.

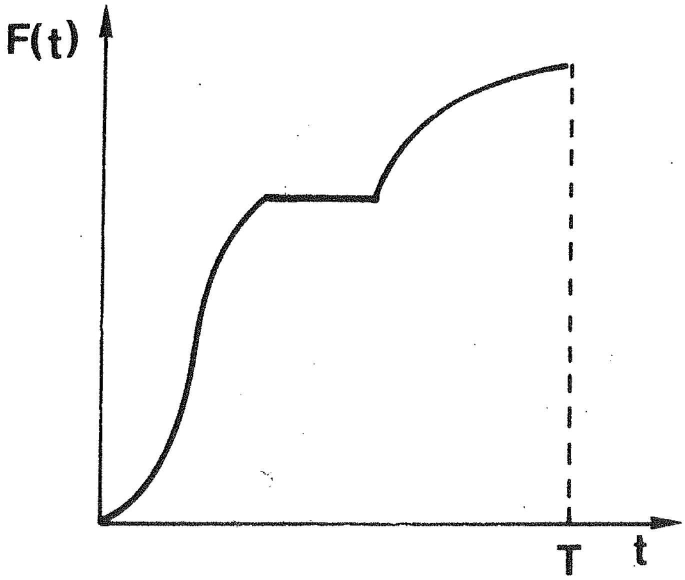


Fig. 3

The functions referring to funds (the $K_i(t)$'s and $L(t)$) measure, for each $t\varepsilon[0, T]$, the amount of the fund which is present (i.e. which is used) in the process in that instant. Fig. 4 illustrates a possible shape of these functions: the fact that zero values alternate with positive values means that the corresponding fund element (a worker, a machine) is not

to deny that this way is totally different from the one which characterizes the flow elements, the funds being the "agents" (see N. Georgescu-Roegen, *The Entropy Law..., op. cit.*, p. 230) of the transformation of flow inputs into flow outputs. The adoption of this point of view also allows for the inclusion in the realm of economic analysis of problems which have been traditionally considered as purely technical ones, and for a more convincing analytical representation of the elements of the process with respect to time. Most of these aspects do not appear in the present use of the flow-fund model, and for a more complete analysis of them, reference is made to the work of Georgescu-Roegen just quoted, chapter IX, and P. Tani, *Analisi microeconomica..., op. cit.*, chapter VII.

continuously necessary in the process. It must be noted that $L(t)$ measures a *number* of workers (and $K_i(t)$ a number of machines of a certain type), while the cumulated labour time used in the process will be measured by a different function, $S(t)$, where:

$$S(t) = \int_0^t L(\tau)\,d\tau.$$

Fig. 5 illustrates the function $S(t)$ corresponding to the function $L(t)$ of fig. 4. Similar functions may be introduced to measure the service of fixed capital funds (again in terms of time of use).

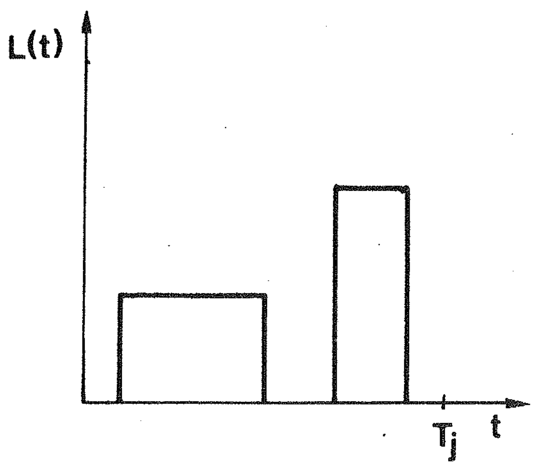


Fig. 4

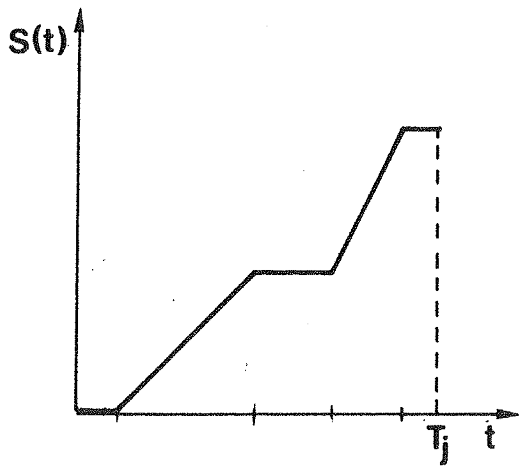


Fig. 5

No special assumption is needed as regards the mathematical properties of these functions. According to the definition, it will be $Q(0) = F_i(0) = 0$; for what follows, it is convenient (i.e. sufficient and not too restrictive) to assume that each function has at most a finite number of discontinuities, and has first derivatives elsewhere. Flow functions are, by definition, non-decreasing functions, but they may have discontinuities; on the contrary, functions of type $S(t)$ are, again by definition, non-decreasing *continuous* functions: this depends on the fact that, while flows may enter the process in a finite quantity even in an infinitesimal interval of time — and this is precisely the type of hypothesis made in discrete time models —, that cannot happen for the service of a fund, which is measured by a time of use of the fund (one hour of labour may be performed by one worker in one hour, or by two workers in half an hour, and so on, but in no case in an infinitesimal time, unless an infinite number of workers were involved).

Time is represented by a continuous variable (more precisely, by a variable that may assume as its value any non-negative real number): however, the assumptions that have been introduced so far allow for a unified treatment of both continuous-time and discrete-time models. In fact, if one wishes to represent the case in which inputs and outputs occur at discrete intervals, it will be sufficient to represent the flow functions by step-functions. The model could be used also with fixed capital represented only by flow functions, simply extending the duration until their productive power has been completely exhausted; however, this method will not be followed in the present paper.

3.3. We assume that each production technique is represented by an *elementary process*, i.e. by a process which gives rise to one unit of product at the end of the duration of the process, the duration being allowed to be different for each process. For each elementary process we have:

$$Q_j(t) = 0, \text{ for each } t \in [0, T_j); \quad Q_j(T_j) = 1 \ (j = 1, \ldots, n),$$

where $[0, T_j)$ is the interval from 0 (included) to T_j (excluded).

The actual production of each commodity is supposed to be performed by an arrangement *in line* of the corresponding elementary process, i.e. by activating elementary processes of the same type, one after another, with a sufficiently short lag so that funds might be used with no idle time.[6] In fig. 6 a stylized representation of an arrangement in line is illustrated, each segment representing an elementary process.

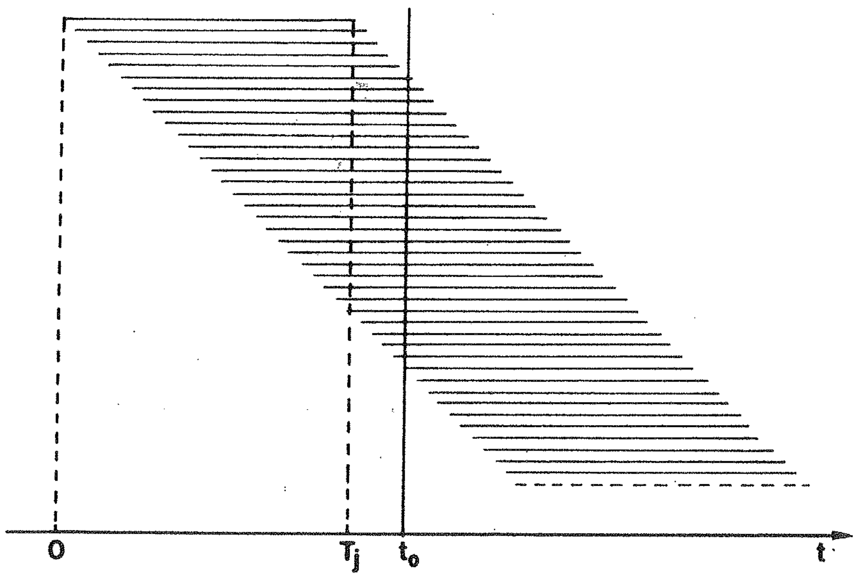


Fig. 6

[6] See N. Georgescu-Roegen, "The Economics of Production", *American..., op. cit.*, p. 6. It must be recognized that the assumption that all processes are arranged in line is a very strong one, especially for non-manufacturing productions.

At time t_0 many processes (twelve, in the figure) are simultaneously active, each one at a different stage. In the case of divisibility, one may let the lag tend to zero, while correspondingly reducing the scale of the processes: at the limit, a *continuous* arrangement in line will be obtained, so that the set of all elementary processes of one type gives rise to continuous uniform flows of inputs and a continuous uniform flow of output. In this case one can also assume that the arrangement in line is performed at a given rate of growth: in this case, the result will be continuous flows of inputs and output growing at the given rate; the amount of each fund will grow at the same rate.

4. STEADY GROWTH RELATIONS IN PRICES AND QUANTITIES IN THE FLOW-FUND MODEL

4.1 Let us now relate the assumptions just made about production processes to the representation of sectorial interdependence. The graph represented in fig. 1 and 2 may again be considered, but now to the arc (x_i, x_j) we shall associate the function $F_{ij}(t)$, $t\epsilon[0, T_j]$ (the meaning of indices i, j is self-evident); to the arc (x_0, x_j) we shall associate the function $L_j(t)$ or the function $S_j(t)$, $t\epsilon[0, T_j]$. In order to represent fixed capital goods, a second graph must be introduced, with functions $K_{ij}(t)$ associated to the arc (x_i, x_j), where $K_{ij}(t)$ measures the amount of fund x_i which is used in the elementary production process of x_j in the instant t.

It can easily be seen that, if no externality in production is to be considered, and if constant returns prevail, at least as far as flows are concerned, then we have:

$$a_{ij} = F_{ij}(T_j); \quad a_{0j} = S_j(T_j) = \int_0^{T_j} L_j(t)\, dt.$$

In fact, in the arrangement in line, each unit of output of the $j.^{\text{th}}$ good will require an amount of the $i.^{\text{th}}$ flow input equal to the quantity of that input which enters the elementary process along its whole duration; this quantity is precisely measured by $F_{ij}(T_j)$. The quantity of labour required for each unit of output of the $j.^{\text{th}}$ good will likewise be measured by $S_j(T_j)$.

4.2. On the other hand, if we start from the above model to describe a stationary or a steady growth state of the productive system, we shall find the following relations (as before, in the text reference is made only to the case of no fixed capital and the general case is considered in the footnotes). The equilibrium price system, at a given uniform (instantaneous) interest rate r, will satisfy the following conditions:

(1) $$p_j = \Sigma_i p_i \int_0^{T_j} e^{r(T_j - t)} dF_{ij}(t) + w \int_0^{T_j} e^{r(T_j - t)} L_j(t)\, dt, \quad j = 1, \ldots, n,$$

where p_j is the price of good x_j, w is the uniform wage rate, and the first integral is a Stiltjes integral.[7]

The equation (1) may be obtained by means of the following argument. In a stationary or steady growth state, the flows of inputs and outputs grow at a constant rate g (possibly zero), and the same is true of the funds. However, this uniform situation is the result of the intertwining of different elementary productive processes, which maintain the features (and so the time profile too) that were attributed to them at the beginning. Consequently, the price of each good must be equal to the cost of the flow inputs and wages, as they appear in the elementary process, valued at the end of the elementary process itself, through the application of the interest rate r. In particular, in (1) $dF_{ij}(t)$ represents the amount of the i.th flow input which enters the production process of good j in the interval $(t, t + dt)$: this amount may be infinitesimal or it may be finite in a certain (finite) number of instants, and zero elsewhere. The first case will occur if the flow of the i.th input is continuous (at least at time t): in this case, the function $F_{ij}(t)$ will have first derivative in t, by the assumptions, and so it will be $dF_{ij}(t) = F'_{ij}(t)\,dt$. If, on the contrary, $dF_{ij}(t)$ is finite (as will be the case for models in which inputs and outputs are assumed to occur only at given points of time, as it is for discrete time models), the Stiltjes integral will be expressed by a sum. The wage rate formally refers to continuous payments: the relevant wage rate for payments at fixed dates can easily be calculated.[8]

[7] If production requires fixed capital, then (1) becomes:

(1′) $$p_j = \Sigma_i p_i \int_0^{T_j} e^{r(T_j - t)} dF_{ij}(t) + w \int_0^{T_j} e^{r(T_j - t)} L_j(t)\, dt + r \Sigma_i p_i \int_0^{T_j} e^{r(T_j - t)} K_{ij}(t)\, dt.$$

It should be remarked that the last addendum in (1′) is the only one referring explicitly to fixed capital, but the flow inputs and the labour fund must contain what is needed for repairs and renewals.

[8] If w is the wage rate for continuous payments and r is the interest rate, then the wage rate W_0 for payments at the beginning of each period of time is given by the following expression:

$$W_0 = w \int_0^1 e^{r(t-1)} dt = w(1 - e^{-r})/r,$$

and the wage rate for payments at the end of each unit period of time is given by the following expression:

$$W_1 = w \int_0^1 e^{r(1-t)} dt = w(e^r - 1)/r,$$

assuming that labour will be performed continuously along the whole duration of the process. If, more realistically, one assumes interruptions, the expressions become more complicated, but it is still possible to express W_0 and W_1 as functions of w and r.

Let us now define:[9]

$$a_{ij}(r) = \int_0^{T_j} e^{r(T_j-t)} dF_{ij}(t), \quad i, j = 1, \ldots, n;$$

(2) $$a_{0j}(r) = \int_0^{T_j} e^{r(T_j-t)} L_j(t)\, dt, \quad i, j = 1, \ldots, n;$$

$$\boldsymbol{A}(r) = [a_{ij}(r)], \quad \boldsymbol{a}(r) = [a_{0j}(r)], \quad \boldsymbol{p} = [p_j].$$

Then system (1) may be arranged in the following matrix form:

(3) $$\boldsymbol{p} = \boldsymbol{p}\boldsymbol{A}(r) + w\boldsymbol{a}(r).$$

In the special case in which every elementary process has all its flow inputs concentrated at time 0, and labour services are uniformly distributed along the whole duration, this duration being one unit of time for all the processes, we shall have:

$$a_{ij}(r) = e^r F_{ij}(1) = (1+\bar{r})\, a_{ij};$$
$$a_{0j}(r) = S_{ij}(e^r-1)/r$$

where $\bar{r} = e^r-1$ is the posticipated annual interest rate equivalent to the instantaneous rate r; system (3) will then become the simpler and more usual expression which follows:

$$\boldsymbol{p} = \boldsymbol{p}\boldsymbol{A}(1+\bar{r}) + W_1 \boldsymbol{a},$$

where W_1 is the wage rate paid at the end of each unit of time (see footnote 8).

From the definitions it is easy to derive the following properties of the matrices and vectors that have just been introduced:

$$\boldsymbol{A}(r) \geq 0; \quad \boldsymbol{a}(r) > 0; \quad \frac{d}{dr}\boldsymbol{A}(r) \geq 0; \quad \frac{d}{dr}\boldsymbol{a}(r) > 0,$$

for each $r > 0$. It follows: if and only if $\boldsymbol{A}(0)$ is a productive matrix, then, for each $w > 0$, equation (3) has a positive solution in $\boldsymbol{p}$ for each $r \varepsilon [0, R]$, where $0 < R \leqslant +\infty$; if the graph of $\boldsymbol{A}$ contains circuits, then $R < +\infty$, i.e., a maximum level $R > 0$ of the rate of interest exists such that, for any non-

[9] For discrete time models, coefficients which are very similar to the $a_{ij}(r)$'s have been used by B. Schefold (who, as far as we know, was the first to introduce them, in "Fixed Capital as a Joint Product and the Analysis of Accumulation with Different Forms of Technical Progress", mimeo, 1974, then published in L. L. Pasinetti (ed.), *Contributions to the Theory of Joint Production*, London, Macmillan, 1978); and, more recently, in neo-Austrian models of the non-integrated type (see C. Belloc, *Croissance économique et adaptation du capital productif*, Paris, Economica, 1980). For a generalization to the case of flow-output, see P. Tani, "Troncabilità dei processi in un modello multisettoriale con capitale fisso intrasferibile", *Rivista Internazionale di Scienze Sociali*, IV, 1978, p. 472.

negative interest rate $r < R$, a non-negative price vector exists which is consistent with that interest rate.

Moreover, for any $r \varepsilon [0, R]$, we have

$$\frac{d}{dr} p > 0$$

so that

$$\frac{d}{dr} \bar{w} < 0$$

i.e. the real wage rate, $\bar{w}$, in terms of any commodity, simple or composite, is decreasing with respect to the interest rate.[10] Therefore, the price system in a steady state has the positive properties of the single output production price system under the more usual assumptions.[11]

4.3. As far as quantities are concerned, again in a steady state, the equilibrium conditions will establish that, for any infinitesimal interval (t, $t + dt$), the flow of output of the i.th good, $q_i dt$, must be equal to the sum of the flows of inputs in different productions and of the flow of consumption. On the other hand, in order to determine the total amount of the flows of inputs in the interval (t, $t + dt$), all the processes which are active at time t must be taken into account. The elementary processes which produce the j.th good and which began at time $t - \tau$ require, during the said interval, $dF_{ij}(\tau)$ unit of good i as input for every unit of output of good

[10] If fixed capital is needed, we shall define also:

$$b_{ij}(r) = \int_0^{T_j} e^{r(T_j - t)} K_{ij}(t); \quad \boldsymbol{B}(r) = [b_{ij}(r)].$$

In this case, (3) becomes:

(3′)
$$\boldsymbol{p} = \boldsymbol{p}\boldsymbol{A}(r) + w\boldsymbol{a}(r) + r\boldsymbol{p}\boldsymbol{B}(r),$$

with

$$\boldsymbol{B}(r) \geq 0, \frac{d}{dr} \boldsymbol{B}(r) \geq 0,$$

so that: if and only if $\boldsymbol{A}(0)$ is productive, then equation (3′) has a positive solution in $\boldsymbol{p}$ for each $r \varepsilon [0, R]$, $0 < R \leq +\infty$; and, for any r in the same interval, we have

$$\frac{d}{dr} \boldsymbol{p} > 0.$$

[11] These positive properties remain valid also with fixed capital (see footnote 10). The model may also be suitably modified to cover the case of "elementary" processes with repeated outputs. However, in this case, the price system loses some of its positive properties and truncation must be allowed in order to restore them (see P. Tani, "Troncabilità...", *op. cit.*).

j which will be available at the end of the process, i.e. at time $T_j + t - \tau$; this output is measured by $q_j(T_j + t - \tau)dt$. Therefore, the input of good i in the interval $(t, t + dt)$ will be obtained from $q_j(T_j + t - \tau)dF_{ij}(\tau)dt$, integrating with respect to τ on the interval $[0, T_j]$ and summing with respect to j. In steady state, at a growth rate g, it will be:

$$q_i(t) = q_j(0)e^{gt},$$

and likewise for $c_i(t)$; therefore, the equilibrium conditions on the quantities will be given by the following equations (where, for convenience, q_i stands for $q_i(0)$ and c_i for $c_i(0)$):

$$q_i e^{gt} = \Sigma_j \int_0^{T_j} q_j e^{g(T_j + t - \tau)} dF_{ij}(\tau) + c_i e^{gt}, \quad i = 1, \ldots, n.$$

Dividing by e^{gt} and changing the integration variable from τ to t, we obtain:

$$(4) \qquad q_i = \Sigma_j q_j \int_0^{T_j} e^{g(T_j - t)} dF_{ij}(t) + c_i, \quad i = 1, \ldots, n.$$

The relevant equations for the fixed capital case are given in the footnote.[12]

In matrix terms, the same system of equations becomes:

$$(5) \qquad \boldsymbol{q} = \boldsymbol{A}(g)\boldsymbol{q} + \boldsymbol{c},$$

where the matrix function $\boldsymbol{A}(\cdot)$ is the same which we found in the price system, and which is defined in (2). Clearly, equation (5) has a non-negative solution for each $g\varepsilon[0, R]$ and for each $\boldsymbol{c} \geq 0$.

With regard to quantities, in the special case in which each elementary process has all its flow inputs concentrated at time 0, and labour services are uniformly distributed along the whole duration, this duration being one unit of time for all the processes, we shall have:

$$\boldsymbol{q} = \boldsymbol{A}\boldsymbol{q}(1 + \bar{g}) + \boldsymbol{c},$$

where $\bar{g}$ is the annual growth rate equivalent to the instantaneous rate g.

[12] With fixed capital, (4) becomes:

$$(4') \qquad q_i = \Sigma_j q_j \int_0^{T_j} e^{g(T_j - t)} dF_{ij}(t) + g\Sigma_j q_j \int_0^{T_j} e^{g(T_j - t)} K_{ij}(t)\, dt + c_i$$

and, in matrix terms:

$$(5') \qquad \boldsymbol{q} = \boldsymbol{A}(g)\boldsymbol{q} + g\boldsymbol{B}(g)\boldsymbol{q} + \boldsymbol{c}.$$

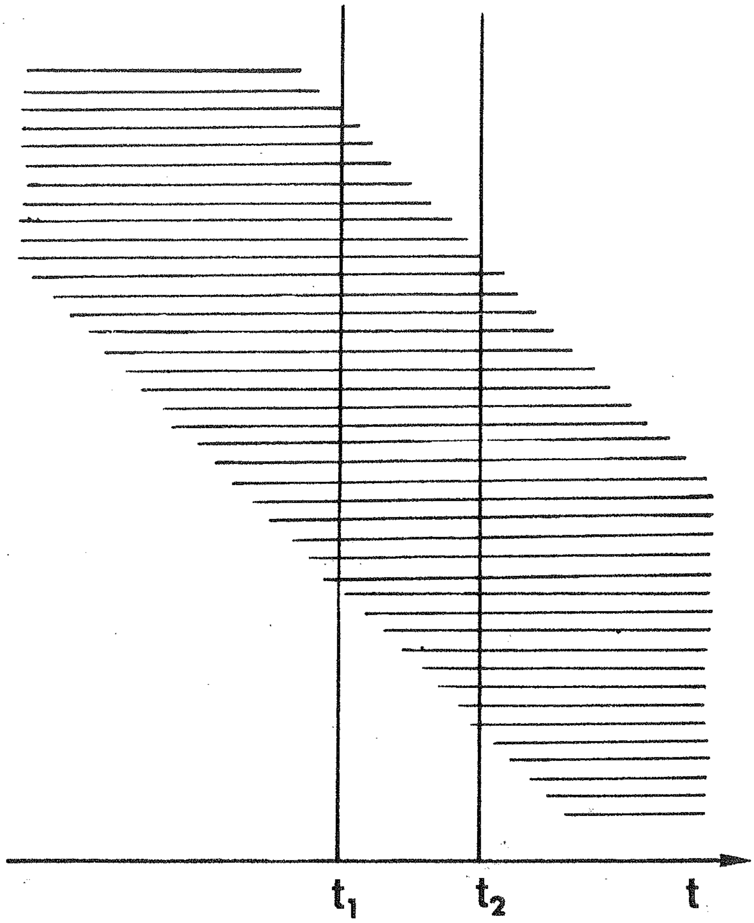


Fig. 7

5. OBSERVED INPUT COEFFICIENTS AND THEIR USE IN QUANTITY AND PRICE SYSTEMS

5.1 Let us now return to the original problem and think of a production system whose processes are of the type we have just described. We defined them as elementary processes and then assumed they were all arranged in (growing continuous) line; as a special case, we assumed that the whole system was in a steady state.[13] It is in this situation that we assume that we observe the system and measure the flows of the input-output table and calculate the input coefficients. In such a situation (see fig. 7), the flow of output x_j over an arbitrary period of time (t_1, t_2) will correspond to the value of the integral

$$\int_{t_1}^{t_2} q_j(t_1)\, e^{g(t-t_1)}\, dt.$$

[13] It must be remarked that the assumption that all the processes are arranged in (growing) line is not equivalent to the assumption of steady state for the entire system: in particular, each sector might have a different rate of growth.

The flow of input x_i entering the production of good x_j during the infinitesimal interval $(t, t+dt)$ will be, as explained above:

$$\int_0^{T_j} q(T_j + t - \tau)\, dF_{ij}(\tau)\, dt;$$

over the interval (t_1, t_2) this flow will then amount to:

$$\int_{t1}^{t2}\int_0^{T_j} q_j(T_j + t - \tau)\, dF_{ij}(\tau)\, dt =$$

$$= \int_{t1}^{t2}\int_0^{T_j} q_j(t_1)\, e^{g(T_j+t-\tau-t1)}\, dF_{ij}(\tau)\, dt =$$

$$= \int_0^{T_j} e^{g(T_j-\tau)}\, dF_{ij}(\tau) \int_{t_1}^{t_2} q_j(t_1)\, e^{g(t-t_1)}\, dt,$$

so that the observed input coefficient $\hat{a}_{ij}$ will correspond to the value of the integral

$$\int_0^{T_j} e^{g(T_j-\tau)}\, dF_{ij}(\tau),$$

which is equal to $a_{ij}(g)$, as defined in (2).[14] It follows that:

(*i*) the coefficients observed are independent of the length of the period over which they are measured;

(*ii*) the production of each commodity may therefore be seen *as if* it were instantaneous: in fact, in each period of time, no matter how long, flows of inputs are transformed into flows of output, according to some ratios which remain constant over time;

(*iii*) these observed coefficients $\hat{a}_{ij}$ will be equal to the $a_{ij}(g)$'s; so $\hat{A} = A(g)$. We have likewise:

$$\hat{a}_{0j} = \int_0^{T_j} e^{g(T_j-t)} L_j(t)\, dt = a_{0j}(g), \quad \text{and so } \hat{\mathbf{a}} = \mathbf{a}(g).$$

[14] With fixed capital, the observed coefficients for the flow inputs remain unchanged. The capital coefficients (i.e., the ratios of the average amount of the capital good over a given period to the flow of output of the period proportioned to one year), calculated on the observed flows and funds, are the following ones:

$$\hat{b}_{ij} = \int_0^{T_j} e^{g(T_j-t)} K_{ij}(t)\, dt = b_{ij}(g).$$

In this case too, the "technical" coefficients, as usually defined, correspond to the $b_{ij}(0)$'s.

5.2. According to (*ii*), we may use observed coefficients as if the productions were performed instantaneously; in equation (5), which defines the total output vector in terms of the exogenous demand for consumption, we may substitute $\hat{A}$ for $A(g)$, so that we have the matrix equation:

$$(6) \qquad q = \hat{A}q + c.$$

The observed matrix will therefore be used for comparative statics results; Leontief's multipliers (the elements of $(I-\hat{A})^{-1}$), based on observed coefficients, can be used whatever the time profiles of the production processes, provided that techniques do not change (which is a normal assumption in input-output exercises) and the system's exogenous growth rate does not change either.

Note that, if $g>0$, we have $\hat{A} = A(g) \geq A(0) = A$, where $A = [a_{ij}]$ is the matrix of "technical" coefficients according to the usual definition. This means that, in the case of a positive rate of growth, the observed coefficients overestimate the "true" technical ones, but, in so far as the system is supposed to maintain the same rate of growth, the observed coefficients give better results than "true" technical ones in comparative statics exercises concerning quantities.

5.3. Thus far the need to know the true time profiles of the elementary processes does not seem to arise. This is not the case when the price system is concerned. In this case, one must know $A(r)$ and, unless one may assume $r = g$, $\hat{A}$ is not a good estimate of $A(r)$; nor is it possible to obtain $A(r)$ from $\hat{A}$, unless special assumptions about the time profiles of inputs are introduced. The fact that the system is in a steady state is not a sufficient condition (unless, again, $r = g$).

Even in a stationary state, an equation of the type

$$(7) \qquad p = p\hat{A}(1+\bar{r}) + Wa$$

has no *a priori* reason to be a good approximation.

Starting from equation (1) and applying some simple transformations to the integrals involved, we get:[15]

[15] From equation (1), integrating by parts, and using the definition of $S(t)$ (see p. 9), we get:

$$p_j = \Sigma_i p_i F_{ij}(T_j) + \Sigma_i p_i \int_0^{T_j} F_{ij}(t)\, e^{r(T_j-t)} r\, dt + w S_j(T_j) + w \int_0^{T_j} S_j(t)\, e^{r(T_j-t)} r\, dt;$$

from which equation (8) follows.

With fixed capital we have:

$$(8') \qquad p_j = \Sigma_i p_i F_{ij}(T_j) + w \int_0^{T_j} L_j(t)\, dt + r\Sigma_i p_i \int_0^{T_j} K_{ij}(t)\, dt + rH_j,$$

where:

$$(9') \qquad H_j = \Sigma_i p_i \int_0^{T_j} e^{r(T_j-t)} F_{ij}(t)\, dt + w \int_0^{T_j} e^{r(T_j-t)} \int_0^t L_j(\tau)\, d\tau\, dt + r\Sigma_i p_i \int_0^{T_j} e^{r(T_j-t)} \int_0^t K_{ij}(\tau)\, d\tau\, dt.$$

(8) $$p_j = \Sigma_i p_i F_{ij}(T_j) + w\int_0^{T_j} L_j(t)\,dt + rH_j,$$

(9) $$H_j = \Sigma_i p_i \int_0^{T_j} e^{r(T_j-t)} F_{ij}(t)\,dt + w\int_0^{T_j} e^{r(T_j-t)} \int_0^{t} L_j(\tau)\,d\tau\,dt,$$

where H_j is the value of the *process fund*, in Georgescu-Roegen's terminology (the Keynesian *working capital*), i.e. the complex of all the semi-finished products, which, in a process arranged in continuous line, has the same composition at any time.[16] In a stationary state, or more generally when each process is arranged in line, production may be considered as instantaneous provided one introduces a new fund, the process fund, i.e. the set of all unfinished products which are constantly present in the process in line. The introduction of this fund does not change the quantity system regarding flows, but, whenever it is $r \neq 0$, it affects the price system: in a steady state, the prices will in fact be equal to the sum of the cost of

[16] See N. GEORGESCU-ROEGEN, *The Entropy Law...*, *op. cit.*, p. 239; J. M. KEYNES, *A Treatise on Money* (1930), London, Macmillan, 1965, vol. I, p. 116 and vol. II, pp. 103 ff. (see also vol. II, pp. 113 ff.), where the nature of the working capital as a fund is clearly stated.

The fact that H_j, as defined in (9) and in (9′), coincides with the value of the process fund, i.e. with the value of all the goods in course of production in the process in line at any time, may be proved in the following way.

Let us consider the initial "queue", when the arrangement in line is started (this queue corresponds to the interval $(0, T_j)$ in fig. 6), i.e. to that initial stage during which the first elementary processes are activated: in this initial stage, the number of processes which are simultaneously performed is growing; it is only starting from T_j that the situation becomes uniform and what has been defined as "process in line" really begins. This initial queue may then be thought of as the production process of the process fund. The value of this process may thus be obtained as the value, at time T_j, of all the inputs which entered in this "initial queue process".

To make this valuation, one may proceed as follows:

1. As far as flow-inputs are concerned, the amount of the flow of the i.th input during interval $(t, t+dt)$ is measured by $F_{ij}(t)$, as may be realised from fig. 6, by noticing that the set of all the infinitesimal stages of the different elementary processes which are performed during that interval represents the interval $(0, t)$ of an elementary process. Therefore, this input will contribute to the value of the process fund to this extent:

$$p_i \int_0^{T_j} F_{ij}(t)\, e^{r(T_j-t)}\,dt;$$

2. As far as funds are concerned, their contribution to the value of the process fund may be obtained by a similar expression, by substituting function $S(t)$ corresponding to the fund for the function $F_{ij}(t)$. In particular, the contribution of labour will be:

$$w\int_0^{T_j} S_j(t)\, e^{r(T_j-t)}\,dt = w\int_0^{T_j}\int_0^{t} L_j(\tau)\,d\tau\, e^{r(T_j-t)}\,dt.$$

Let F be the value of the process fund; it will then be:

$$F = \Sigma_i p_i \int_0^{T_j} F_{ij}(t)\, e^{r(T_j-t)}\,dt + w\int_0^{T_j}\int_0^{t} L_j(\tau)\,d\tau\, e^{r(T_j-t)}\,dt + \Sigma_i r p_i \int_0^{T_j}\int_0^{t} K_{ij}(\tau)\,d\tau\, e^{r(T_j-t)}\,dt,$$

from which it can be verified that F coincides with H_j, as defined in (9′).

flow inputs, of wages and of the interest on the value of the working capital (and, possibly, also of the fixed capital). On the other hand, the value of H_j is not generally connected in a simple way with the total cost of flow inputs, since it depends crucially on the time of the payments made for these inputs, and thus on their time profile.

5.4. Ways out of this problem, and, in particular, ways of relating the value of H_j to the total cost of flow inputs, are obtained by assuming special time profiles for the inputs of each process. The most usual one is based on the following hypotheses:

a) all the elementary processes have the same duration;

b) all the inputs enter the process at time 0 of the elementary process.

As we showed above (see p. 13), under these assumptions the usual equation for the price system holds: however, these are very strong assumptions about time profiles, especially if the list of goods and the length of the observed period must both be considered as arbitrarily given.

Another way out is to assume that, independently of the duration of the elementary process and of the time profile of the inputs in it, all payments are made at given dates (e.g., at the end of each month). This assumption is a reasonable one for some inputs (labour, electricity), but it cannot easily be accepted as a general assumption. More precisely, it is difficult to accept that the flow of payments is totally independent of the time structure of the elementary process; or, what amounts to the same thing, that the working capital has nothing to do with the time profile of the process.

A third way out is to assume that H_j does not change much with r. It is certainly not necessary to recall the theoretical problems connected with this assumption; but, of course, it may be of use in the application of input-output analysis to problems of price variation (e.g. the inflationary results of the rise in the price of some primary input).

Two final remarks may be useful. First, the above analysis in no way denies the possibility of introducing generalised assumptions about the time profiles of the production processes in growth models and in long-run price analysis: on the contrary, it shows how this can be made in a fairly general way. What the analysis is meant to show is that the introduction of these arbitrary time profiles, while not preventing the ordinary use of input-output matrices of the observable type, does create serious problems as to their use in the analysis of prices and distribution.

The result just stated — and this is the second remark — may seem inconsistent with the well-known duality results, according to which propositions concerning quantities may always be turned into relevant propositions concerning prices: but the inconsistency is only apparent.

Indeed, what is usually required from the input-output models as far as quantities are concerned is to determine how outputs change (or must change) when exogenous demand changes. What the price system is meant for is the analysis of how prices, and the real wage, change when the rate of interest changes, which is a different, and much more demanding question. The corresponding (dual) problem as to the quantity system would be the analysis of how quantities change when the rate of growth changes: this problem would encounter difficulties of the same type as those we pointed out for prices.[17]

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[17] In any case, one difference will remain: while the present (uniform) rate of growth is incorporated in the observed coefficient (see above, p. 17), the same is not true for the present rate of interest.