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Linear Joint Production Models: Prelude to a Reassessment of the Classical Legacy (*Value, Equilibrium, and Disequilibrium*)

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Introduction

Piero Sraffa's "Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory",[1] which was published nearly thirty years ago, was at the origin of the debate on joint production. Although not fully original,[2] this debate expanded in the 1970s and is still an issue in the 1980s.

The problems raised by the treatment of joint production are often difficult and several aspects still remain unsolved. One controversial issue in this debate is that of the relationship between the labour theory of value and the theory of production prices. Ian Steedman[3] made use of some of the properties of joint production to attempt to disprove the labor theory of value. He argued as follows: since production prices can exist with negative "values", how can prices be interpreted, following Marx's analysis, as *forms* of value? Other less controversial issues question the consistency of the theory of production prices in itself, independently of the relationship between values and prices: existence of production prices, relation between supply and demand, and the choice of the technology.

The purpose of the present study is to discuss the fruitfulness of this theoretical debate on joint production. Our general view is that it is not joint production *per se* which can open new avenues. Rather we believe that the importance of the analysis of joint production is that is represents a test of excellence. Any fuzziness in the definition of fundamental notions

* We thank Christian Bidard for helpful discussions and Mark Glick for his help in the translation of this text into English.

[1] Cf. P. Sraffa, *Production of Commodities by means of Commodities, Prelude to a Critique of Political Economy*, Cambridge, Cambridge University Press, 1960

[2] Cf. H. Kurz, "Joint Production in the History of Economic Thought", *Metroeconomica*, vol. 38, 1986, pp. 1-37.

[3] Cf. I. Steedman, "Positive Profits with Negative Surplus Value", *The Economic Journal*, vol. 85, March 1975, pp. 114-123.

or any lack of understanding of basic mechanisms are exposed when joint production is considered. Interestingly, the development of the debate on joint production has been at the origin of a kind of revolution in the approach to the classical theory of prices and economic phenomena in general (by classics, we refer to A. Smith, D. Ricardo and, by extension, to K. Marx). The notion of a classical equilibrium, in which the supply and demand sides of the economy are considered, has been re-established, at least implicitly. This metamorphosis has already been realized in several works concentrating on the choice of the technology. However, it is our opinion that, behind this restoration of the classical conception of equilibrium, the most promising concept, that of the "classical conception of disequilibrium" (an expression by which we mean the analysis that the classics gave of the competitive process), lies backstage.

In what follows, we will not attempt to reproduce the various stages in which the debate on joint production evolved. Successively considered will be the three dimensions of the classical legacy: The labour theory of value (part I), The classical conception of equilibrium (part II), and The classical conception of disequilibrium (part III). We will try to show in each case how the debate on joint production surprisingly contributed to the opening of new perspectives in these fields.

I. The Labour Theory of Value

This first part is devoted to the problem of the determination of values. In section A, the term value is used in its traditional sense. The specific properties of joint production are opposed to that of single production. In section B, the notions of *individual* and *market* values are substituted for the traditional concept. How this substitution solves the positivity/negativity puzzle is demonstrated in section C, where the concept of *reductivity* is introduced and used in the formulation of the conditions of existence.

A. *Traditional Values in Single and Joint Production*

With the following notation:

- n Number of goods (also equal to the number of processes)
- Λ Vector $(n \times 1)$ of values
- L Vector $(n \times 1)$ of labour inputs (strictly positive)
- A Matrix $(n \times n)$ of material input coefficients
- B Matrix $(n \times n)$ of output coefficients

values, in the traditional sense of the term, are defined as a strictly positive solution of equation (1) for single production, and equation (2) for joint production:

$$\Lambda = A\Lambda + L \tag{1}$$

$$B\Lambda = A\Lambda + L \tag{2}$$

The existence of values in the case of single production is guaranteed (a necessary and sufficient condition) by the *productivity* of the technology. With the notation:

z Vector $(1 \times n)$ of levels of activity

$\leq$ $v \leq w$ means $\forall i \;\; v^i \leq w^i$

$<$ $v < w$ means $\forall i \;\; v^i \leq w^i$ and $v \neq w$

$\ll$ $v \ll w$ means $\forall i \;\; v^i < w^i$

productivity can be defined as follows:

Definition 1: Productivity. *A technology is productive, if a set of levels of activity* z *can be found for which all components of the net product are strictly positive*:

$$\exists z > 0 \quad \textit{such that} \quad z(B-A) \gg 0$$

Productivity is usually viewed as a natural condition, and the fact that a non-productive technology leads to the determination of negative "values" in single production is not considered as a problem or an objection to the labour theory of value.

In joint production the productivity of the technology is neither a necessary nor a sufficient condition for the existence of values. In the literature, no alternative condition is generally acknowledged as a substitute for productivity. Moreover, the existence of production prices is not a sufficient condition for the existence of values. Thus, it is possible to define technologies which yield production prices yet have no values. With the notation:

p Vector $(n \times 1)$ of production prices

consider the following example in which the technology is productive, and production prices exist, but values do not exist:

$$A = \begin{pmatrix} 5 & 2 \\ 1 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 7 & 5 \\ 2 & 2 \end{pmatrix} \Rightarrow \Lambda = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \quad \text{and} \quad \frac{p}{w} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad \text{if} \quad r = \frac{2}{3}$$

A consequence of this property is that positive profits can coincide with a negative rate of exploitation. The "Fundamental Marxian Theorem",[4]

[4] Cf. M. Morishima, *Marx's Economics, a Dual Theory of Value and Growth*, Cambridge, Cambridge University Press, 1973.

which states that a positive rate of exploitation is a necessary condition for the existence of a positive rate of profit, does not hold in joint production.

These properties appear to dramatically illustrate how damaging the study of joint production could be to the labour theory of value. If the demonstration above were not questioned, the consideration of joint production would appear as a crucial contribution to the progress of economic theory. Fortunately, this is not the case.

B. *Individual and market (or average) values*

An important contribution to the debate on the existence of values in joint production was formulated by Michio Morishima.[5] Morishima suggested that the mode of computation of values must be modified in joint production, *since a commodity can be produced in different processes of production*. He defined the notion of the "true value" of a commodity, as the *minimum* amount of labour necessary for its production. A further step was taken by Peter Flaschel in 1983,[6] when he pointed out that Marx's distinction between individual and market values should be applied to the case of joint production, for the reason already mentioned by Morishima (a commodity can be produced in different processes). The *individual* value of a commodity corresponds to the labour time necessary for its production under its specific conditions. The *market* (or average) value of a commodity refers to the average (and not minimum) of individual values (Marx's "socially necessary labour time"). The limitation of Flaschel's approach was, however, that he did not directly apply the formalism of individual and market values to the case of joint production, but resorted to a procedure of disaggregation of joint processes. This approach is possibly relevant with respect to empirical studies, but it is alien to the labour theory of value as such.

The first direct application of the distinction between the two types of values, individual and market values, to joint production was realized by the present authors.[7] Below we briefly recall the general line of argument, starting with Marx's definitions, and the results obtained.

Marx's distinction is stated as follows:

> "*Besides this, however, there is always* a market value *(of which more later), as distinct from the individual value of particular commodities produced by the different*

[5] Cf. M. Morishima, "Positive Profits with Negative Surplus Value: a Comment", *The Economic Journal*, vol. 86, September 1976, pp. 599-603.

[6] Cf. P. Flaschel, "Actual Labour Values in a General Model of Production", *Econometrica*, vol. 51, n. 2 1983, pp. 435-454.

[7] Cf. G. Dumenil and D. Levy, "Labour Values and the Inputation of Labour Contents", Paris, CEPREMAP, 1986; G. Dumenil and D. Levy, "Values and Natural Prices Trapped in Joint Production Pitfalls", *Zeitschrift für Nationalökonomie, Journal of Economics*, vol. 47, n. 1 1987, pp. 15-46.

producers. The individual value of some of these commodities will stand below the market value (i.e., less labour time has been required for their production than the market value expresses), the value of the others above it. Market value is to be viewed on the one hand as the average value of the commodities produced in a particular sphere, and on the other hand as the individual value of commodities produced under average conditions in the sphere in question, and forming the great mass of these commodities".[8]

This new approach required a number of transformations in the traditional formalism: 1) There is no reason for the number of processes to be equal to that of commodities, 2) It becomes necessary to distinguish a single commodity when it is produced by two different processes, 3) Commodities which are purchased as inputs by capitalists do not transfer to the product their individual value but only their average or market value. Using the notation:

m Number of processes

i Index of a good (superscript)

j Index of a process (subscript)

A^i_j Input in good i of process j

A Matrix $(m \times n)$ of material input coefficients

B^i_j Output in good i of process j

B Matrix $(m \times n)$ of output coefficients

L Vector $(m \times 1)$ of labour input coefficients

$\bar{\Lambda}$ Vector $(n \times 1)$ of market values

Λ^i_j Individual value of commodity i as produced by process j

the value equations can be written as follows:

$$\underbrace{\sum_{i=1}^{n} B^i_j \Lambda^i_j}_{\text{Individual values of the outputs}} = \underbrace{\sum_{i=1}^{n} A^i_j \bar{\Lambda}^i}_{\text{Market values of the inputs}} + \underbrace{L_j}_{\text{Newly incorporated labor}} \qquad \textit{for } j = 1, \ldots, m \tag{3}$$

$$\underbrace{\bar{\Lambda}^i}_{\text{Market values}} = \underbrace{\sum_{j=1}^{m} z_j B^i_j \Lambda^i_j \sum_{j=1}^{m} z_j B^i_j}_{\text{Average of individual values}} \qquad \textit{for } _i = 1, \ldots, n \tag{4}$$

[8] Cf. K. MARX, *Capital*, vol. III, New York, First Vintage Book Edition, 1981.

The first equation concerns one process (j), while the second, a single good (i). Five results followed:

1. *Individual and market values, as positive solutions of equations* (3) *and* (4), *are not determinate in a unique manner*. The number of equations is $n + m$, and the number of unknowns is equal to n plus the number of elements of B which are different from zero. This indeterminacy does not change the fact that, for conceptual reasons, only the concept of individual and market values should be applied to the case of joint production.
2. The value of the net product of the economy is always determinate: $z(B - A)\bar{\Lambda} = zL$. This result is important if one adheres to the Duménil-Foley interpretation of the Transformation Problem,[9] which refers to this aggregate.
3. As we will show in the next section, the conditions (the non-reductivity of the technology) for the existence of individual and market values are very weak.[10]
4. When they exist, values determined by the traditional equations (with $m = n$ as in equation 2) correspond to a particular case of individual and market values, for which all individual values of the same commodity are equal and, consequently, are also equal to the market value. However, the requirement of the identity of the labour times necessary for the production of the same commodity within different processes is irrelevant with respect to the theory of value.
5. As far as empirical computations are concerned, it is possible to avoid the indeterminacy of values, using the procedures of disaggregation such as those suggested by Flaschel.[11] In a recent paper,[12] we analyze the properties of such disaggregations and show that several rules, different from that proposed by Flaschel, can be used, depending on the problem considered.

The substitution of the notions of individual and market values for the traditional approach represents a first example of the manner in which the treatment of joint production can rekindle an old distinction, which had been unduly forgotten.

[9] Cf. G. Dumenil, *De la valeur aux prix de production*, Paris, Economica, 1980; G. Dumenil, "The So-called 'Transformation Problem' Revisited: A Brief Comment", *Journal of Economic Theory*, vol. 33, n. 2 1984, pp. 340-348; D. Foley, "Value of Money, the Value of Labour Power and the Marxian Transformation Problem", *Review of Radical Political Economy*, vol. 14, 1982, pp. 37-47; see also A. Lipietz, "The So-called 'Transformation Problem' Revisited", *Journal of Economic Theory*, vol. 26, 1984, pp. 59-88.

[10] In particular in the example introduced above (cf. A above), for $z = (1, 1)$, a set of individual values is $\Lambda_1^1 = 3/7$, $\Lambda_1^2 = 1/5$, $\Lambda_2^1 = 3/7$, and $\Lambda_2^2 = 1$. The corresponding market values are $\bar{\Lambda}^1 = 3/7$ and $\bar{\Lambda}^2 = 3/7$.

[11] Cf. P. Flaschel, "Actual Labour Values...", *op. cit.*

[12] Cf. G. Dumenil and D. Levy, "Labour Values...", *op. cit.*

C. *Reductivity*

The labour theory of value applies to disequilibrium situations as well as equilibria. Therefore, a condition for the existence of values must be more general than the existence of classical equilibrium (the existence of prices of production). The labour theory of value is valid even in cases where prices are different from production prices. In the general case, several processes are used for the production of the same commodity, and the rate of profit is not uniform. (The labour theory of value should also apply to the case of a square technology, *i.e.*, with $m = n$, in which production prices do not exist for a given rate of profit.)

It is a mere coincidence that the notion of productivity, in square single production, defines the conditions for the existence of values. In fact, another condition also exists, which is equivalent to productivity in the case of square single production models, which, in our opinion, is far more adequate to the problem considered, and can be generalized to the case of single production with $m > n$ or to joint production. It is the condition of *non-reductivity* of the system. Reductivity is defined as follows:

Definition 2: Reductivity. *A technology is reductive if a set of levels of activity, z, can be found for which all components of the net product are negative or zero*:

$$\exists z > 0 \quad \textit{such that} \quad z(B-A) \leq 0$$

This condition was introduced for the first time by G. Dumenil[13] and later formalized by G. Dumenil and D. Levy.[14] In a linear formalism such as the one used in this study, the interpretation of *reductivity* is straightforward. In a reductive system, it is possible to produce (using a given process of production or a linear combination of such processes) and to obtain a net product (*Material outputs – Material inputs*) negative in all of its components. Such an action cannot be classified as productive: it is negative production, or "reduction". If a technology is reductive the computation of values leads quite naturally to the determination of the *positive value* of a *negative net output*, which is abusively interpreted as the negative value of a positive bundle of commodities, after simply changing the signs.

In single production, with $m \geq n$, the non-reductivity (or "productivity") of the *average technology* is a necessary and sufficient condition for the positivity of individual and market values. In joint production no simple necessary and sufficient condition for the existence of individual and market values can be formulated. However, *the non-*

[13] Cf. G. Dumenil, *De la valeur..., op. cit.*, p. 45.
[14] Cf. G. Dumenil and D. Levy, "Values and Natural Prices Trapped...", *op. cit.*

reductivity of the system is a sufficient condition for the existence of values. The proof of this proposition is given in our 1986 and 1987 papers on this topic,[15] with slightly different assumptions.

The assumption that the economy is non-reductive is a very weak assumption. This is evident from the definition of non-reductivity, equivalent to definition 2, *in the price system*:

Definition 3: A Second Approach to Reductivity. *A technology is non-reductive, if a set of prices,* p, *can be found for which the prices of all net products are positive*:

$$\exists p \gg 0 \quad \textit{such that} \quad (B-A)p \gg 0 \tag{5}$$

It is difficult to include, as part of the technology actually in use, processes of production such that, for *any* set of prices, the price of the output will not cover the cost of material inputs, leaving no remuneration to labour or capital.

The Fundamental Marxian Theorem follows from the above condition. If production prices exist, they obviously insure that $(B-A)p = Lw + r(Ap + Lw)$ is strictly larger than zero, provided that w or r is also larger than zero. Thus, individual and market values exist. This result guarantees, in turn, that the rate of exploitation is also positive.

In sum, the formulation of the conditions for the positivity of values in a model with joint production can be retrospectively seen as a fruitful enterprise. It seriously contributed to the substitution of a new, more adequate, condition (non-reductivity) for an inadequate one (productivity), which cannot be generalized.

II. Classical Equilibrium

In this second part of this study, we will discuss the issue of classical equilibrium: *production prices* and the related *proportions of output*. The first section addresses the issues of prices and outputs — taken separately — as has often been the case in the classical literature (section A). We call this partition "the dichotomy between prices and outputs". In the following sections, the two aspects of equilibrium (prices and outputs) are treated simultaneously. We first introduce the definition of a classical equilibrium (section B), then the problems of the existence, uniqueness or multiplicity of equilibrium are discussed, together with a number of other related properties (section C).

[15] Cf. G. Dumenil and D. Levy, "Labour Values...", *op. cit.* (Proposition 2); G. Dumenil and D. Levy, "Values and Natural Prices Trapped...", *op. cit.*

A. *The dichotomy between prices and outputs*

In subsection 1, we shall briefly recall the properties of single production relative to *prices* and restate the conditions for the existence of production prices in joint production, for the case of a square technology, independently of outputs. Subsection 2 introduces the discussion of the relation between *supply and demand* in the general case of joint production. Subsection 3 shows how this problem led some researchers to formulate new technological conditions which would guarantee the existence of the properties obtained in single production (a nostalgic approach to science). Lastly, subsection 4 marks the break with the traditional dichotomy between prices and quantities, and introduces the notion of the simultaneous treatment of prices and quantities which was originally characteristic of the classical conception of equilibrium.

1. Production Prices in Single and Joint Production

In single production, production prices are defined by one of the following two equations:

$$p = (Ap + Lw)(1 + r) \quad \text{or} \quad p = Ap(1 + r) + Lw$$

The choice depends on the assumptions which are made concerning the payment of wages. We prefer the first expression, but very few results depend on this choice. In a similar manner, one obtains in joint production:

$$Bp = (Ap + Lw)(1 + r) \quad \text{or} \quad Bp = Ap(1 + r) + Lw$$

In single production, the productivity condition guarantees that prices of production exist for a range of values of the rate of profit from zero to a maximum rate of profit, R, (or from zero to a maximum rate of wages). Beyond this limit, no set of production prices can be found. In joint production no such simple rule can be established. Production prices can only be defined for several separate segments of values of the rate of profit (the same problem is posed if the rate of wages is given).

In earlier works,[16] we attempted to formulate conditions, generalizing the notion of reductivity. It is possible to extend this notion (cf. definition 2) to define a property which we call "Radical r-Domination" (this definition is strictly equivalent to that of reductivity when $r = 0$):

Definition 4: Radical r-Domination. *We say that radical r-domination exists in a technology whenever it is possible to define some vector of levels of activity* $z > 0$ *such that* $z(B - (1 + r)A) \leq 0$.

[16] Cf. G. Dumenil and D. Levy, "The Unifying Formalism of Domination Value, Price, Distribution and Growth in Joint Production", *Zeitschrift für Nationalökonomie, Journal of Economics*, vol. 44, n. 4, 1984, pp. 349-371.

We further define:

Definition 5: Nonradical r-Domination. *We say that nonradical* r*-domination exists in a technology whenever it is possible to define two vectors of levels of activity,* z > 0 *and* z′ > 0, *such that*:

$$z(B-(1+r)A) \leq z'(B-(1+r)A) \quad \text{and} \quad zL \geq z'L$$

with, at least, one strict inequality among the $n+1$.

The absence of (radical and nonradical) r*-domination in a technology is a necessary and sufficient condition for the existence of production prices, with a uniform rate of profit equal to* r. However, this does not imply the uniqueness of the solution (generally obtained if $m = n$).[17]

These conditions in terms of domination are connected with our earlier analysis, since radical r-domination directly prolongs the concept of reductivity, and since nonradical r-domination, in turn, prolongs radical r-domination.

2. All-Productivity

In single production, prices can be determined independently of the value of outputs. This corresponds to the traditional approach in the literature of classical inspiration. This dichotomy between price and quantity variables is the result of a specific property of single production, which Bertram Schefold called "all-productivity":[18] Any bundle of commodities can be obtained as net output of the system. In the same study Schefold also introduces the notions of "all-engaging technologies" and "r-all-engaging technologies". Both types are all-productive technologies. The former group satisfies the condition: $(B-A)^{-1} \gg 0$, and the latter: $(B-(1+r)A)^{-1} \gg 0$.

In all-productive systems, the levels of activity and the value of outputs in each commodity can be studied independently of prices. For example, if final demand is modeled as a given vector D, and productive demand determined by the requirements for balanced growth at a given rate ρ, then

[17] Such "domination" has been previously discussed in the literature. Several authors have referred to properties similar to rath of domination. Cf. L. Rampa, "Valori lavoro e spreco di lavoro nei modelli di produzione congiunta", *Giornale degli Economisti e Annali Di Economia*, vol. XXXV, n. 9-10 1976, pp. 601-621; E. Wolfstetter, "Positive Profits with Negative Surplus Value: a Comment", *The Economic Journal*, vol. 86, 1976, pp. 864-876; L. Filippini, "Positività dei Prezzi e Produzione Congiunta", *Giornale degli Economisti e Annali Di Economia*, vol. XXXVI, gennaio-febbraio 1977, pp. 91-99; C. Filippini and L. Filippini, "Two Theorems on Joint Production", *The Economic Journal*, vol. 92, 1982, pp. 386-390; Y. Fujimori, "Outputs, Values and Prices in Joint Production", University of Josuai, Japan; Y. Fujimori, *Modern Analysis of Value Theory*, Berlin, Springer Verlag, 1982; M. Lippi, *I prezzi di Produzione*, Bologna, Il Mulino, 1979. However, the distinction between radical and nonradical domination had, to our knowledge never been presented.

[18] Cf. B. Schefold, "Multiple Product Techniques with Properties of Single Product Systems", *Zeitschrift für Nationalökonomie, Journal of Economics*, vol. 38, n. 1-2 1978, pp. 29-53.

it is always possible to compute the appropriate levels of activity: $Bz = zA(1+\rho) + D$ or $z = D(B - A(1+\rho))^{-1}$ which is larger than zero, if $\rho < R$. (Of course, if D is a function of p, it is also the case for z).

Under general circumstances, joint production technologies are not all-productive, only a subset of the possible proportions of demand can be realized. All-productive technologies posses the properties of single production in many respects, hence the title of Schefold's article: "*Multiple product techniques, with properties of single product systems*".[19] This absence of all-productivity raises obvious serious difficulties concerning the equalization of supply and demand.

A first reaction to these problems was to be faithful to the traditional dichotomy between prices and quantities and to search for the conditions which would guarantee the traditional properties (cf. subsection 3). However, the view that prices and quantities should be treated separately is no longer dominant in the literature (cf. subsection 4).

3. Conditions on the Technology

Sufficient conditions for a system to be r-all-engaging can be found in the literature.[20]

Consider a square technology (A, B, L). If and only if right and left generalized eigenvectors exist, and are unique and positive for a rate of profit R, *i.e.*, if:

$$\exists z \gg 0 \quad \text{such that} \quad z(B - (1+R)A) = 0$$

and

$$\exists p \gg 0 \quad \text{such that} \quad (B - (1+R)A)p = 0$$

then, $\exists r_0$ such that the system is r-all-engaging for $]r_0, R[$:

$$\forall r \in]r_0, R[, \quad (B - (1+r)A)^{-1} \gg 0$$

Such a system possesses all the properties characteristic of single production for $r \in]r_0, R[$: 1) Production prices exist, 2) Wage-prices, p^i/w, are increasing functions of the rate of profit, 3) For a given numeraire, the rate of wages is a decreasing function of the rate of profit on the factor price frontier, and 4) All proportions of demand can be satisfied (as net product).

A discussion of these problems can be found in a paper by G. Abraham-Frois and E. Berrebi,[21] for the case of $m = n = 2$.

[19] Cf. B. SCHEFOLD, "Multiple Product Techniques...", *op. cit.*

[20] Cf. C. Bidard, "Is von Neumann Square?", *Zeitschrift für Nationalökonomie, Journal of Economics*, vol. 46, 1986, pp. 401-419; B. SCHEFOLD, "Multiple Product Techniques...", *op. cit.*

[21] Cf. G. ABRAHAM-FROIS and E. BERREBI, "Taux de profit minimum dans les modèles de production", in C. BIDARD (ed.), *La production jointe, Nouveaux débats*, Paris, Economica, pp. 211-299.

4. Classical Equilibrium

There is now a rather broad agreement[22] on at least three basic issues, concerning classical equilibrium:

1. Demand must be explicitely taken into account.
2. The appropriate framework of analysis is that of long-term equilibrium, with a uniform rate of profit and balanced growth.
3. In an equilibrium, no process exists whose rate of profit is larger than the uniform rate of the processes which are used (Cost-Minimizing System).

It is important to notice the exact nature of the transformation of the perspective, from the study of production prices to that of a general equilibrium of the *von Neumann-Morishima* type.

A classical equilibrium differs in several respects from a Walsarian equilibrium:[23]

1. Classical equilibrium is a *long-term* equilibrium with mobility of capital, which must not be confused with an Arrow-Debreu equilibrium — a *short-term* equilibrium, with given production functions and, thus, no mobility of capital.
2. A second aspect of the long-term perspective in the classical equilibrium is that the equilibrium is independent from the initial endowments in reproducible goods.
3. In the classical perspective, the treatment of the labor market and monetary phenomena do not replicate that of the commodity market (in particular, in the analysis of production prices, the situation of distribution is considered given).

B. *The definition of equilibrium*

This section divides into six subsections — from a presentation of the general guidelines (subsection 1) to the definition of equilibrium itself (subsection 5) and the comparison of this definition with others (subsection 6).

[22] Cf. C. BIDARD and R. FRANKE, "On the Existence of Long-term Equilibria in the Two-class Pasinetti-Morishima Model", *Ricerche Economiche*, vol. XLI, n. 1 1987, pp. 3-21; R. FRANKE, "Joint Production and the Existence of Cost-minimizing Systems", Bremen, Universität Bremen; R. FRANKE, "Some Problems Concerning the Notion of Cost-minimizing Systems in the Framework of Joint Production", *The Manchester School*, vol. LIV, n. 3 1986, pp. 298-307; N. SALVADORI, "Switching in Methods of Production and Joint Production", *The Manchester School*, vol. LIII, n. 2 1985, pp. 156-178; N. SALVADORI and I. STEEDMAN, "Joint Production Analysis in a Sraffian Framework", Mimeo, 1988; B. SCHEFOLD, "The Dominant Technique in Joint Production Systems", *Cambridge Journal of Economics*, vol. 12, 1988, pp. 97-123.

[23] Cf. G. DUMENIL and D. LEVY, "The Classicals and the Neoclassicals, A Rejoinder to Frank Hahn", *Cambridge Journal of Economics*, vol. 9, 1985, pp. 327-345.

1. General Guidelines

The definition of equilibrium derives from a particular view of the behavior of the economy in a vicinity of this equilibrium. In the definition of an equilibrium it should be clear that one refers to the equilibrium of a dynamic process (a sequential process) which occurs in a situation of disequilibrium and whose outcome is the tendency toward equilibrium. For this reason, this process could be called the "operating" process of the equilibrium.

The relation between equilibrium and disequilibrium is more or less explicit in the literature. As a result of the classical inspiration, the general notion of the *mobility of capital* is common to this literature. This explains why a general agreement has been possible on the idea of "Cost-Minimizing System": If the rate of profit were larger in a process which is not used (a manifestation of disequilibrium), capital would adopt this process. The idea that an equilibrium refers to an operating process is, however, more general. It should not be restricted to the migration of capital, but also concerns the other aspects of the model: Demand, distribution, etc.

It is along such guidelines that the controversial issues should be dealt with:

1. How should demand be modeled?
2. Should "free goods" (and free disposal) be considered?
3. Is it more appropriate to assume a given rate of wages or rate of profit?

Because of the number of unclarified issues, the study of classical equilibrium is one of the most appealing aspects of the analysis of joint production, for future research. As in the case of the theory of value and production prices, we believe that the rigor which the treatment of joint production requires from the economist which investigates its properties can lead to important clarifications, destroy a number of artificial barriers, and stress the actual differences between the classical and neoclassical perspectives. Below we will not attempt to precisely answer the questions raised, but instead simply list the problems, suggest directions, and formulate hypotheses.

2. The Framework of Analysis

In the definition of equilibrium, the technology (A, B, L) and the situation of distribution are given. A difficulty lies in this latter element. One must assume a given wage: either a real wage, or a nominal wage expressed in a numeraire.[24] In our opinion, it is not consistent with the

[24] The difference between the two conceptions is that workers are supposed to consume their real wage, whereas the assumption of a nominal wage expressed in a numeraire only fixes the purchasing power of workers, independently of their actual choice.

classical conception of equilibrium to assume a *given* uniform rate of profit, since this approach cannot be generalized to the treatment of disequilibrium. The classical, and very realistic, conception is that capitalists, aware of the price of their inputs (physical inputs and labor power), choose among the various uses of their capital on the basis of the existing — unequal — rates of profit (possibly the expectations formed on the observation of past rates). The equilibrium average rate of profit is a virtual magnitude as long as convergence has not been achieved. It is not a price, as is the case for the rate of wages.

3. Demand

In order to study equilibrium it is necessary to make precise assumptions concerning the formation of demand. Two very different types of problem are involved: The *proportions* of demand, and its general size or *dimension*.[25]

- *Proportions*

By "proportions" of demand, we mean the relative value of its various components. The point at issue concerning the definition of equilibrium is whether one should allow a certain degree of flexibility in the modeling of demand. For reasons which go back to the work of P. Sraffa[26] himself, and because of the neoclassical sympathy with flexible demand, the students of production prices still tend to reject flexibility. Our view is that a limited degree of flexibility should be considered. It is clear that final consumers will never consume machine tools, even if the price of these machines tends to diminished in comparison to that of consumption items. However, we believe that any actual fall toward zero of the price of a commodity which is already demanded will stimulate its demand (for final or productive consumption). How much flexibility should be authorized cannot be determined *a priori*. In the construction of models the investigation should, therefore, consider various cases, from the absence of flexibility to a reasonable degree of flexibility. A useful, although not quite realistic, assumption is that the fall in the price of a good i toward zero pushes its demand toward infinity:

$$\text{If} \quad p^i \to 0, \quad \text{then} \quad D^i \to \infty \tag{6}$$

In this assumption the tendency for demand to growth infinitely must be seen as a substitute for "a sufficient growth".

[25] We borrow this terminology from our work on the stability of capitalism (cf. for example G. Duménil and D. Levy, "Stability and Instability in a Dynamic Model of Capitalist Production" (abridged version), in W. Semmler (ed.), *Competition, Instability and Nonlinear Cycles*, Lecture Notes in Economics and Mathematical Systems, Berlin, Springer Verlag, 1986, pp. 132-169).

[26] P. Sraffa, *Production of Commodities...*, *op. cit.* Sraffa assumes that, at a given instant, demand is given. He calls this demand vector "Requirements for use" (cf. for example note 2, § 50). However, nothing is said of the manner in which this demand is determined.

- *Dimension*

By the "dimension" of demand, we mean its general level, independent of its composition. When equilibrium is reached, the price of aggregate demand must obviously be equal to that of aggregate supply, and to income. The problem is whether this equality is satisfied only when equilibrium prevails, or whether it must also hold in a disequilibrium. In this latter case, this equality is, in fact, an identity (called "Walras' Law"[27]). A simple and common assumption in models of classical inspiration which deal with the existence of equilibrium (and not with its stability) is to assume that *income is equal to production and is actually spent at each period*, and to abstract from phenomena related to disequilibrium and business cycle.

As is well known, in Keynesian models, demand is not equal to total income *ex ante*, but is a function of income. Equality is obtained when the Keynesian "equilibrium", which is not related to the full or "normal" utilization of resources, prevails. The concept of the classical long-term equilibrium is irrelevant in this perspective. In our models, where we study the stability of capitalism, demand can also differ from total income in disequilibrium. The operating process which insures that equilibrium is the "normal" classical equilibrium (with a normal utilization of productive capacities, no inflation and, of course, as is the case for all equilibria, *demand = income = output*) is the issuance of money and credit.[28]

Even in a simple framework which abstracts from disequilibrium in dimension, an important question remains opened, that of the rate of accumulation. Which agents accumulate and how much:

1. The most simple assumption was made in the original von Neumann model[29]: The consumption of workers is included in the technology, capitalists do not consume and accumulate all profit.
2. Following Morishima,[30] it is possible to add complexity to this model. Capitalists share their profits between consumption and accumulation in a given proportion. If s denotes their propensity to save, then one obtains the famous relation $\rho = sr$.
3. A more complex case is one in which "workers" are also allowed to save. As is well known, the relation $\rho = sr$ is conserved.[31]

[27] Cf. R. Franke, "Some Problems...", *op. cit.*

[28] This equilibrium is only stable under given conditions (cf. G. Dumenil and D. Levy, "The Macroeconomics of Disequilibrium", *Journal of Economic Behaviour and Organization*, vol. 8, 1987, pp. 377-395).

[29] Cf. J. von Neumann, "A Model of General Economic Equilibrium" (1938), *The Review of Economic Studies*, vol. 13, 1945-46, pp. 1-9.

[30] Cf. M. Morishima, *Equilibrium, Stability and Growth*, Oxford, Clarendon Press, 1964.

[31] Cf. L. Pasinetti, "Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth", *Review of Economic Studies*, vol. 29, n. 2 1961, pp. 267-279, for the original discussion in a one commodity model; for a treatment of this topic in Morishima model cf. C. Bidard and R. Franke, "On the Existence of Long-term Equilibria...", *op. cit.*

4. Free Disposal

In the expression *free disposal*, "disposal" refers to a procedure in which the excess production of a commodity ($S^i - D^i$) is destroyed. Under general assumptions, this disposition requires the use of a specific process, similar to a productive process, but whose function is the opposite. The input of such processes of disposal would have a negative equilibrium price. "Free" disposal means that there is no cost to this destruction. In the discussion of disposal two types of goods must be distinguished:

- *True Waste*

True waste consists of outputs which are never demanded at any price. It is possible to eliminate these items from the formalism and to include the cost of their disposal in the description of the productive processes in which they are obtained. Therefore, the consideration of a specific process modeling this disposal is not necessary.

- *Goods Demanded but Produced in Excess*

Consider now the case of a commodity, i, whose demand is strictly larger than zero but strictly smaller than its supply ($0 < D^i < S^i$). In this case the assumption of free disposal is crucial, because it avoids the consideration of negative equilibrium prices. This assumption is, however, often rejected as unrealistic.[32]

In both the cases of true wastes or other goods, we still believe that "free disposal" can be used as a first approximation (cf. examples below in II.C.3). In our opinion the most realistic way of dealing with this problem is, however, to assume that a price which tends toward zero stimulates demand (cf. assumption 6) and allows the equalization between supply and demand (and, thus, the problem of disposal disappears). In the traditional example of meat and wool, even if only one breed of sheeps exists, a relative price of meat in comparison to wool will be determined in such a way that no excess of any item exists.

5. Definition

In this subsection three definitions of equilibrium are given depending on the option concerning demand and disposal.

With the notation:

R^n_+ the set of vectors $v \geqq 0$

R^n_{++} the set of vectors $v \gg 0$

[32] For example: "[...] but free disposal is not a realistic assumption", N. Salvadori and I. Steedman, "Joint Production Analysis...", *op. cit.*

consider a technology (A, B, L) and the rules which define the formation of final demand $d(z, p, w)$.

1. *Equilibrium with Equality between Supply and Demand*

An equilibrium is a multiplet

$$(z, p, r, w) \in R_+^m \times R_{++}^n \times R_+ \times R_{++},$$

with strictly positive prices, such that the following conditions are satisfied:

a. There is no superprofit

$$r_j \leq r \forall j \quad \text{or} \quad 1 + r_j = \frac{B_j p}{A_j p + L_j w} \leq 1 + r$$

or, as is traditionally written:

$$B_j p - (1 + r)(A_j p + L_j w) \leq 0 \quad \text{for} \quad j = 1, \ldots, m$$

b. A process with a rate of profit smaller that the regulating rate is not used.

If $r_j < r$, then $z_j = 0$ or:

$$z(B - (1 + r)A)p = zLw(1 + r)$$

c. Supply is equal to demand:

$$z(B - A) = d(z, p, w)$$

2. *Equilibrium With Disposal*

When processes of disposal are included in the technology, total demand (final and productive) can be smaller than supply. Equilibrium ($S = D$) is insured by the consideration of the "demand" emanating from processes of disposal. Conditions a., b., and c., above are conserved. Prices can be negative and equilibrium is a multiplet in

$$R_+^m \times R^n \times R_+ \times R_{++}.$$

3. *Equilibrium With Disposal*

When disposal is free, it is possible to give another definition of equilibrium, which is equivalent to the above (equilibrium with disposal).

Processes of disposal are not included in the technology. Prices are positive or equal to zero. The first two conditions above are conserved, but c. is replaced by d. and e.:

d. Supply is larger than, or equal to, demand:

$$z(B - A) \geq d(z, p, w)$$

e. A good produced in excess is a free good.

If $S^i > D^i$, then $p^i = 0$ or:

$$z(B - A)p = d(z, p, w)p$$

With the exception of all-productive systems, the third requirement of an equality between supply and demand (in the strict sense, *i.e.*., without considering demand emanating from processes of disposal) is difficult to obtain, unless some flexibility is introduced in the modeling of demand (cf. assumption 6).

Note that these definitions of equilibrium do not imply any definite option concerning distribution. One option would be to fix w and to define equilibrium as the triplet (z, p, r). Another option would be to fix r. As we mentioned earlier, the first option should be preferred to the second one, but from the point of view of the definition of equilibrium, no such choice need to be made.

6. A Comparison

As was shown above, the problem of the definition of equilibrium in joint production can be posed in different ways. A first distinction concerns the assumption of free disposal and its consequences on the relation between supply and demand. A second issue concerns the meaning which is given to the equality between the price of supply and demand. Either this equality is viewed as a property of equilibrium or as an identity. This distinction is important in the proof of the existence of equilibrium, but it does not really correspond to a different conception of equilibrium. In the table below, we compare four approaches to these problems:

$S \geq D$ $pS \equiv pD$ C. Bidard and R. Franke[33]	$S = D$ $pS \equiv pD$ R. Franke[34]
$S \geq D$ B. Schefold,[35] N. Salvadori[36]	$S = D$ N. Salvadori[37]

The upper left corner of this table corresponds to the *von Neumann-Morishima* type of approach. The lower right corner corresponds to Sraffa's perspective.

In the paper by C. Bidard and R. Franke,[38] a rate of growth, ρ, can

[33] Cf. C. BIDARD and R. FRANKE, "On the Existence of Long-term Equilibria...", *op. cit.*
[34] Cf. R. FRANKE, "Joint Production...", *op. cit.*; R. FRANKE, "Some Problems...", *op. cit.*
[35] Cf. B. SCHEFOLD, "The Dominant Technique...", *op. cit.*
[36] Cf. N. SALVADORI, "Switching in Methods...", *op. cit.*, § IX.
[37] Cf. N. SALVADORI, "Switching in Methods...", *op. cit.*
[38] Cf. C. BIDARD and R. FRANKE, "On the Existence of Long-term Equilibria...", *op. cit.*

be associated to each value of the rate of profit, r (since $\rho = sr$). In the paper by B. Schefold,[39] $d(z, p, w) = pzA + d$, in which d (a vector) and ρ are given (as well as r). In § IX of the paper by N. Salvadori (lower left corner of the table),[40] $d(z, p, w) = d$ in which d is a given vector.

C. Existence and related properties

In this section, we first discuss the issue of the existence and uniqueness of equilibrium (subsection 1). The next subsection consists of a few remarks concerning the properties of this equilibrium with respect to the factor price frontier (subsection 2). Then, we present two examples (subsection 3). Last, a few remarks are proposed concerning the relevance of technologies in which $m = n$ (subsection 4).

1. Existence and Uniqueness

Below we consider three theorems of existence of equilibrium which correspond respectively to the three different definitions:

1. Free Disposal (and Free Goods):[41]

   Consider a given technology (A, B, L), a given rate of profit r, a given demand function (such that income is identically equal to the price of demand). Under the assumption of r-productivity, *i.e.*, if:

   $$\exists \bar{z} \in R^m_+ \quad \text{such that} \quad \bar{z}(B - (1 + r)A) \geqslant 0 \tag{7}$$

   (and if $L \geqslant 0$), then an equilibrium with free disposal and positive prices

   $$(z, p, w) \in R^m_+ \times R^n_+ \times R_{++}$$

   exists.

2. No Free Disposal:[42]

   The assumption of *free disposal* in the previous theorem conditions the positivity of equilibrium prices ($p^i \geq 0$). In R. Franke, these prices can be negative. Outputs with such negative prices are interpreted as "waste". Condition 7 is replaced in the theorem of existence by:

   $$\forall c \in R^n_+,\ \exists \bar{z} \in R^m_+ \quad \text{such that} \quad \bar{z}(B - (1 + r)A) = c$$

   which can be called "r-all-productivity".

[39] Cf. B. Schefold, "The Dominant Technique...", *op. cit.*

[40] Cf. N. Salvadori, "Switching in Methods...", *op. cit.*, § IX.

[41] Cf. C. Bidard and R. Franke, "On the Existence of Long-term Equilibria...", *op. cit.*

[42] Cf. R. Franke, "Joint Production...", *op. cit.*; R. Franke, "Some Problems...", *op. cit.*

3. Equality between Supply and Demand and Assumption 6 on Demand:[43]

With the assumptions of theorem 1 above, and the assumption that $D^i \to \infty$ when $p^i \to 0$, then an equilibrium with strictly positive production prices and equality between supply and demand exists.

Concerning the uniqueness of equilibrium, no general theorem exists.

None of these conditions can be considered satisfactory. A serious deficiency, common to these three theorems, is that r is taken as given, instead of w. R. Franke[44] raises the problem of a given w, but with the same treatment of demand as in the second theorem above. One can find in a paper by P. Medvegyev[45] the proof of a theorem of existence, with w given. Two criticisms can be addressed to this analysis: 1) The positiveness of ρ and r is not insured, 2) No maximum rate of wages is considered explicitly.

2. Equilibrium and the Factor Price Frontier

On the basis of a square ($m = n$) technology and a given numeraire, one can define a wage-profit relation $w(r)$ or $r(w)$. If $m > n$, such a wage-profit relation can be defined for each square technology (combining n processes among the m). All these relations can be graphically represented in the orthant defined by the rate of wages and the rate of profit. The curves can have several intersects. For each value of the rate of wages, a particular technology yields the highest rate of profit. The broken line formed by the various segments corresponding to these specific technologies is called the factor price frontier. Segments are separated by switch points.

A first difference between single and joint production formalisms is that the rate of profit of the switch points between the various technologies on the frontier depends on the numeraire. Moreover, for a given rate of profit, the technology which is used at equilibrium does nt necessarily yield the largest rate of wages (or the largest r, for a given w): Technologies which could satisfy demand and yield r and w larger than those characteristic of equilibrium can also exist.

In a specific sense of optimality (different from the neoclassical notion[46]), an equilibrium is not necessarily an optimum.

[43] Cf. C. Bidard and R. Franke, "On the Existence of Long-term Equilibria...", *op. cit.*
[44] Cf. R. Franke, "Joint Production...", *op. cit.*
[45] Cf. P. Medvegyev, "A General Existence Theorem for von Neumann Economic Growth Models", *Econometrica*, vol. 52, n. 4 1984, pp. 963-974.
[46] In the neoclassical perspective, it is the final consumption of the owners of the enterprises which is considered in the definition of the optimum and not the rate of profit of enterprises.

3. Two examples

Because of the complexity of the properties introduced above, it can be helpful to consider numerical examples. The two examples below illustrate the multiplicity of equilibria, and contrast a model in which demand is rigid and equilibrium is obtained with free disposal (example 1), with a model in which final demand of capitalists is flexible (example 2).

- *Example 1*

Technology is as follows:

$$A = \begin{pmatrix} 0 & 30 \\ 20 & 14 \end{pmatrix} \quad B = \begin{pmatrix} 15 & 48 \\ 38 & 31 \end{pmatrix} \quad L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

All profits are consumed ($s = \rho = 0$) and, with the notation:

W Total wages

Π Total profits

(1, 1) Numeraire and given proportions of consumption

Demand is given by:

$$d(z, p, w) = (W + \Pi)\frac{1}{p^1 + p^2}(1, 1)$$

One has:

$$d(z, p, w)p = W + \Pi = z(B - A)p$$

For $w = 6$, three equilibria exist — one with two processes, and two with one process:

1. The levels of activity are such that: $z_1/z_2 = 1/3$. Prices are: $p^1 = p^2 = 1/2$. The rate of profit is: $r = .5$.
2. Only the first process is used: $z_2 = 0$. Prices are $p^1 = 1$ and $p^2 = 0$. The rates of profit are: $r_1 = 1.5$ and $r_2 = 6/13$. Thus, $r_2 < r_1$.
3. Only the second process is used: $z_1 = 0$. Prices are $p^1 = 0$ and $p^2 = 1$. The rates of profit are: $r_1 = 1/3$ and $r_2 = .55$. Thus, $r_1 < r_2$.

- *Example 2*

In this second example, the assumptions are as in example 1, except demand emanating from capitalists which is flexible. Final worker demand is still proportional to the basket (1, 1). Thus, workers consume (6, 6) for $w = 6$. Demand from capitalists is proportional to $(d^1(p), d^2(p))$, with:

$$d^i(p) = 17 - 6\frac{p^i}{p^1 + p^2}$$

As in the case of example 1, three equilibria exist:

1. The first equilibrium is unchanged.
2. Only the first process is used: $z_2 = 0$. Prices are $p^1 = 5/6$ and $p^2 = 1/6$. One can verify that $r_1 > r_2$. Demand from capitalists is proportional to (3, 4). Supply is equal to demand.
3. Only the second process is used: $z_1 = 0$. Prices are $p^1 = 55/138$ and $p^2 = 83/138$. One can check that $r_2 > r_1$. Again the equality between supply and demand can be insured.

The two equilibria with free disposal in example 1 are approximations of the corresponding equilibria with flexible demand in example 2. Note that it is not necessary to assume that $D^i \to \infty$ if $p^i \to 0$, to obtain equilibria with positive prices.

4. Square Technologies

The issue of the relevance of square technologies ($m = n$) has been debated in the literature. Square technologies can be obtained by setting $n - m$ prices to zero and deleting from the formalism all goods whose price is equal to zero. In the first example above, the three equilibria are square. For the first equilibrium: $m = n = 2$, for the two others: $m = n = 1$.

Equilibria with $m < n$ can exist if a certain degree of flexibility in demand is allowed. (In the second example above, for the second and third equilibria, $m = 1$ and $n = 2$.) In this case, prices are not fully determinate on the basis of the traditional set of equations ($m - n$ degree of freedom exist). This indeterminacy is avoided by imposing the equality of supply and demand. In our opinion, this case is the more realistic one, since technologies are not necessarily square.

III. Classical disequilibrium

This last part introduces a few remarks concerning the classical analysis of disequilibrium. More specifically, we shall consider the process of formation of production prices within competition, in joint production formalisms. The first section states the nature of the issue (section A). In the second section we discuss the issue of the stability of classical equilibrium (section B). In the last section we present a few remarks concerning the resistance to the introduction of disequilibrium in the classical literature (section C).

A. *Disequilibrium and the competitive process*

In a first subsection, we shall describe the place of disequilibrium in the classical legacy (subsection 1). The second subsection contrasts the analysis of the stability of the competitive process with the study of the convergence of an algorithm such as that proposed by Sraffa (section 2).

1. Disequilibrium in the Works of the Classics

In the works of the classics and Marx,[47] one can find descriptions of the mechanisms which are supposed to insure the convergence of market prices toward production prices.[48] This analysis deals with the workings of a decentralized market economy, in which individual agents make individual decisions and behave in a specific manner which reflects their social nature.

An important characteristic of this analysis of the classics is that the economy is considered to be in a disequilibrium state, in the sense, that markets do not clear (inventories of unsold commodities or rationing can exist), productive capacities are not necessarily used at their full or normal level, capital has not been allocated properly among industries, and prices are not equal to production prices. To model these mechanisms in a way faithful to the classics, it is necessary to consider the behaviour of economic agents in a disequilibrium.

Smith, Ricardo, and Marx clearly distinguished between market prices and natural prices, or production prices, *i.e.*, between prices in disequilibrium and prices which prevail in a long-term equilibrium, when the migration of capital has proportioned productive capacities to "effective demand", in Smith's words, or "social needs", in Marx's terminology.

2. The Competitive Process and Algorithms

In his study of the choice of the technology, Sraffa[49] refers to an algorithm. The steps are the following, with a given rate of profit:

1. A square technology is considered. The corresponding production prices and the rate of wages are computed, for the given rate of profit.
2. The rates of profit of the processes which are excluded from this technology are verified using the production prices and rate of wages associated with this technology.
3. If the rate of profit of an excluded process is larger than r, this process is introduced in the technology and another process is set aside.

[47] Cf. A. Smith, *The Wealth of Nations* (1776), London, Dent and Son Ltd, 1964, ch. 7; D. Ricardo, *The Principles of Political Economy and Taxation* (1817), London, Dent and Son Ltd, 1960, ch. 4; K. Marx, *op. cit.*, ch. 10.

[48] It is well known that these mechanisms create a tendency towards a long-term equilibrium which is constantly affected by perturbations, and that the result of this convergence is a gravitation of market prices and outputs around this equilibrium. Moreover, the target is constantly shifting due to technological change, changes in wages, in tastes, etc.

[49] Cf. P. Sraffa, *op. cit.*

of this analysis was, to our knowledge, our 1983 study,[52] later generalized in other studies of ours.[53] Little work has been devoted, however, to the specific problems which are encountered in a joint production model. P. Flaschel and W. Semmler consider joint production, but actually model what should be called a "classical *tâtonnement*".[54] A number of interesting remarks can also be found in the work of B. Schefold.[55]

2. Problems in obtaining Convergence

In the present subsection, using the first example in II.C.3, we briefly illustrate the type of problems which arise in the study of the competitive process under the assumption of joint production. In order to study the stability of three equilibria, we shall make the following assumptions:

1. There is only one price for each commodity on the market. The dynamic of prices is such that: *Supply of good* 1 > *Demand of good* 1 or *Supply of good* 2 < *Demand of good* 2 result in a diminished ratio of prices p^1/p^2. (Recall that total supply is equal to total demand).
2. We abstract from the problem of the possible rationing of buyers and assume that excess supply is freely disposed of at each period (no inventory).
3. The rates of profit are computed on the basis of the technology and prices, independently of what is actually sold.
4. Capital is fully utilized.
5. Capital movements are based on relative profitability ($z_1/z_2 \nearrow$ if $r_1 > r_2$ and conversely).

Under these assumptions, one can verify that the two equilibria, in which only one process is activated, are stable. But the equilibrium with two processes activated is unstable. This latter property is due to the fact that

[52] Cf. G. DUMENIL and D. LEVY, "La concurrence capitaliste: un processus dynamique", in J. P. FITOUSSI and P. A. MUET (eds.), *Macrodynamique et déséquilibres*, Paris, Economica, 1987., pp. 127-154.

[53] Cf. G. DUMENIL and D. LEVY, "The Competitive Process in a Fixed Capital Environment: a Classical View", The Manchester School, vol. LVII, n. 1 1989, pp. 35-57; G. DUMENIL and D. LEVY, "The Dynamics of Competition...", *op. cit.* (see the appendix to this study for a survey of the literature, but a particular mention sould be made of L. BOGGIO, "On the Stability of Production Prices", *Metroeconomica*, vol. XXXVII, n. 3, pp. 241-267; L. BOGGIO, "Stability of Production Prices in a Model of General Interdependence", in W. SEMMLER (ed.), *op. cit.*); G. DUMENIL and D. LEVY, "The Stability of Long-term Equilibrium in a General Disequilibrium Model", Paris, CEPREMAP, n. 8717, 1987.

[54] Cf. P. FLASCHEL and W. SEMMLER, "Classical and Neoclassical Competitive Adjustment Processes", *The Manchester School*, n. 1 1987, pp. 13-37.

[55] Cf. B. SCHEFOLD, "The Dominant Technique...", *op. cit.*

starting with equilibrium prices ($p^1 = p^2 = 1/2$), a small increase of the first price results in the following chain of events:

$$\left(\frac{p^1}{p^2} \nearrow\right) \Rightarrow (r_1 < r_2) \Rightarrow \left(\frac{z^1}{z^2} \searrow\right) \Rightarrow (S^1 \searrow \quad \textit{and} \quad S^2 \nearrow) \Rightarrow \left(\frac{p^1}{p^2} \nearrow\right)$$

and the distance from equilibrium is increased.

This example illustrates the fact that stability is not necessarily insured. Parenthetically, the first equilibrium in the example, which is unstable, is probably that which fits best with the traditional conception of production prices in joint production (the two processes are activated, there is no free good).

In the general case, an open question is whether at least one stable equilibrium exists. There is no simple answer to this problem. The answer would obviously depend on the model considered.

C. *Who's afraid of disequilibrium?*

The analysis of disequilibrium represents a new opportunity for modern researchers to restore and develop the classical analysis. It is clear, however, that this investigation still faces serious resistances. This reluctant attitude toward disequilibrium is particularly evidenced by the refusal to deal with inventories and capacity utilization:

1. Concerning inventories — the obvious manifestation of disequilibrium on the market (inventories are nothing else than the difference between supply and demand) — it is striking to notice that they are considered in only a few classical models. Apart from our models, to which we referred throughout this study, one can only find two papers by L. Boggio[56] and a study by R. Franke[57] (which was directly inspired by our work) dealing with these issues. Reviewing one of our articles, I. Steedman[58] is actually puzzled by our reference to inventories.
2. The case of capacity utilization rates of fixed capital is even worse. To our knowledge, it has been completely ignored in the literature of classical inspiration. More attention has been paid to the various vintages of fixed capital, than to the treatment of capacity utilization rates. The traditional modeling of fixed capital using a joint production formalism cannot claim to be general in this respect.

[56] Cf. L. Boggio, "On the Stability...", *op. cit.*; L. Boggio, "Stability of Production Prices...", *op. cit.*

[57] Cf. R. Franke, "Production Prices and Dynamical Processes of the Gravitation of Market Prices", Dissertation, Universität Bremen, Bremen.

[58] Cf. I. Steedman, "Natural Prices, Differential Profit Rates and the Classical Competitive Process", *The Manchester School*, June 1984, pp. 123-140.

The consideration of disequilibrium is of crucial interest. It conditions, not only the study of the stability of classical long-term equilibrium, but also the analysis of the stability of capitalism, Ricardo's "states of distress", Marx's "crises", Keynes' stagnation, the permanent features of capitalism, from its early to its modern stages of development.

Conclusion

It is not possible to limit the contribution of the classics — in the sense of A. Smith, D. Ricardo and, by extension, K. Marx — to the three fields considered in this study: 1) Value, 2) Long-term equilibrium (production prices and associated outputs), and 3) The analysis of the mechanisms which, in a context of disequilibrium, are able to insure the gravitation around this equilibrium. However, these three groups of issues define three prominent aspects of economic theory which must be revived in their pre-Walrasian state of development, and then brought to a new stage of maturity.

Surprisingly enough, the sophisticated and very abstract analysis of linear models with joint production contributed to this restoration. The reason for this paradoxical outcome is that the difficulties met in the treatment of joint production required a strict reassessment of basic classical notions. Although this restoration is still quite incomplete, several examples of this have been given in the present study:

1. In the analysis of values, the negativity problem rekindled the old distinction of K. Marx between individual and market values, which must be applied whenever a single commodity can be produced in different manners (as is the case in joint production). Moreover, the traditional "productivity" condition, which is alien to the labour theory of value and cannot be generalized to joint production formalisms, has been replaced by that of "nonreductivity" which refers to the concept of the embodiment of labour in a straightforward manner (in a reductive system, labour can be expanded to "reduce" the inputs, a negative production) and can be generalized to the case of joint production.
2. Concerning prices and outputs, the examination of the various stages through which the debate developed is quite revealing. As long as single production models were used, it was obvious that prices could be analyzed independently of quantities (levels of activity and outputs). This property was even interpreted as a distinguishing feature of the classical analysis. When these issues were addressed in joint production models, this view of the specificity of the classical model hampered the progress of the analysis: researchers were

concerned with the formulation of conditions which would insure the existence of properties identical to those which prevail in single production. In a progressive manner, and rather unobtrusively, the idea emerged that demand had to be explicitly considered. Rigid proportions of demand are still a common assumption and a neoclassical flavour still attaches to the flexibility of these proportions in relation to prices. However, simultaneously, the perspective evolved from that of the computation of prices of production to that of the definition of a classical equilibrium.

3. The investigation of the classical analysis of disequilibrium represents a new barrier to be overthrown, which we consider crucial in the establishment of the actual differences between the classical and neoclassical perspectives (rigid demand does not make a classic). The modeling of disequilibrium processes defines a first field in itself, but the consideration of joint production has not played any particular role in this investigation, to date. It is in an indirect manner that the consideration of disequilibrium in joint production models can illuminate the ongoing controversy concerning equilibrium (its definition and existence). Disequilibrium is a crucial element in our ability to choose among various approaches to equilibrium. Conditions of existence of equilibrium which take the uniform rate of profit as given cannot survive in a perspective which relates equilibrium to the (disequilibrium) processes which are supposed to insure its stability.

As far as joint production itself is concerned, the definition and conditions of existence of classical equilibrium constitutes the issue on which the debate should concentrate. The study of disequilibrium represents, however, the most promising field opened for future investigation.

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