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Joint Production in Sraffa: Some Open Issues

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I shall try to give an orderly answer to the questions posed by *Political Economy*. Naturally, I shall be very brief on those points that I think are by now beyond discussion, and concentrate on more controversial issues. In particular, as regards the fourth question, I shall discuss to some extent: (1) simultaneous determination of prices and quantities in a general joint-production system; (2) gravitation of market prices towards prices of production; we shall see that a serious difficulty is inherent in joint-production. I shall not perhaps be complete in quotations and acknowledgements. But I hope this may be excused in a paper aiming above all at highlighting open problems.

1. Sraffa opened Chapter X of *Production of Commodities by Means of Commodities*, Fixed Capital, by stating that: "The interest of Joint Products does not lie so much in the familiar examples of wool and mutton, or wheat and straw, as in its being the genus of which Fixed Capital is the leading species". This seems to me rather unsatisfactory as a representation of reality. In fact, industry provides a lot of examples of joint production which are far more interesting and important than those rather "preindustrial" cases, while not being of the fixed capital kind. In any case, after some hesitation, around the end of the seventies it became clear that general joint production was, after all, tractable. Actually, in the last ten years general joint production has drawn increasing interest and, as I shall argue in Sections 2 and 3, fairly conclusive results have been given for some of the most important problems.

Having implicitly responded to the first question, I shall add, as regards the third, that single production should be considered as no more than a useful benchmark for the general case. For instance, all the so-called paradoxes of joint production are such only insofar as comparison with single production is concerned. And yet, the attempt to give a solution to the paradoxes has led to very important insights into joint production (see Section 2).

Lastly, before I turn to the second question, I should point out that one important case of joint production has, as far as I known, been left almost untouched. For most multi-product industries the proportion of outputs may be varied continuously within a given range. Consider, for instance, the following two-product industry with flexible-output proportions and a linear constraint:

$$ap(1+r) + lw = hp_1 + kp_2$$
$$h + k = 1.$$

Now, it is immediately clear that competition will force p_1 and p_2 to fulfill:

$$ap(1+r) + lw = p_1$$
$$ap(1+r) + lw = p_2.$$

On the basis of such a simple example one might think that pervasiveness of joint production is overstated, because multi-product flexible-output industries are equivalent to a single production subsystem. But is the example representative enough? In particular, is linearity a reasonable general assumption? And in non-linear cases, under which assumptions would the "tangent system" behave well?[1]

2. As regards the second question, it must be remembered that in the general joint-production case analyzed by Sraffa: (1) negative prices may occur at a rate of profit r, even when r is well below the maximum rate of profit (even for $r = 0$, when prices are proportional to quantities of labour embodied); (2) the wage-profit relationship can fail to be decreasing for some value of r in the relevant range and for some commodity used as measure of prices. Other problems arise relative to the results obtained in the single-production case, but I will not try to be exhaustive.

I must confess I never fully understood the importance of the second problem. Once it has been made clear that the rising stretch of the wage-profit curve has nothing to do with any adding-up of quantities rising independently from one another, its possible occurrence in a joint production system endowed with sensible economic properties is a mere fact, without any particular positive or negative connotation.

I shall therefore concentrate on the first "paradox" and set out in this Section a short illustration of the solution which has been given, while in the next Section I shall indicate some open problems.

A short premise is necessary. What I am going to say may be interpreted

[1] On this point I only know of a few lines in B. Schefold, "On Changes in the Composition of Output", *Political Economy*, 2, 1985, p. 136.

in two different ways: either we are assuming constant returns to scale, or all changes of produced quantities we shall consider are to be taken as virtual, like when the standard system is obtained from the real system. I shall return to this point in the next Section.

Now, start with an $n \times n$ joint-production system: call it S. Assume, for simplicity, that the net product of the system is entirely consumed: call y the net-product vector. Then consider all systems obtained from S via elimination of some of the industries, with the condition that the remaining ones, taken together, are still able to produce y, maybe overproducing some of its components: call such subsystems $S_1, S_2, \ldots, S_m$ (including S itself). It has been shown that if r lies below R, where R is the maximum rate of steady growth of S, then prices are non-negative for at least one of subsystems S_i. More precisely, if A and B are, respectively, the input and outputs matrices of S, while l is the vector of labour quantities, then, for $0 \leq r < R$, a vector p of non-negative prices and a vector q of non-negative activity levels exist and fulfill the following relationships:

$$\begin{aligned} Bp &\leq A(1+r)p + l \\ B'q &\geq A'q + y, \end{aligned} \qquad \text{(PQ)}$$

where $p_i = 0$ if the i-th commodity is overproduced, $q_i = 0$ if the i-th industry makes a loss (prices are measured in labour). The non-zero q_i's identify the selected subsystem, let it be S_k. It is important to remark that excluded methods yield losses with respect to prices p, i.e. prices prevailing in S_k. For this reason chosen subsystems, which will not be unique in general and will vary with r, have been called Cost-Minimizing Systems.[2] Thus the paradox fades away if we consider that a joint-production system "hides" subsystems, which are smaller but nonetheless vital (i.e. able to produce at least y as net product), and if we allow these subsystems to come out when necessary, that is when the original system happens not to be a Cost-Minimizing System.

Lastly, as the results just reported generalize without any difficulty to non-square systems (A, B, l), then the same analytical framework is suitable for analyzing the choice of techniques.

3. If we interpret the changes in activity levels considered above as merely virtual, then the resort to subsystems S_i may be considered as a suggestive explanation of negative prices, but no more. Such an attitude

[2] Various authors have contributed to the "contamination" of the Sraffa system with inequalities. I shall recall: L. PUNZO, "Labour-Values in Single Product and Joint Product Systems", *Economic Notes*, 1, 1978; B. SCHEFOLD, "On Counting Equations", *Zeitschrift für Nationalökonomie*, 4, 1978; M. LIPPI, *I prezzi di produzione*, Bologna, 1979; N. SALVATORI, "Existence of Cost-Minimizing Systems within the Sraffa Framework", *Zeitschrift für Nationalökonomie*, 42, 1982.

may be supported on the grounds of Sraffa's strong statement, right at the beginning of *Production of Commodities by Means of Commodities*, where he asserted: "No changes in output and (at any rate in Parts I and II) no changes in the proportion in which different means of production are used by an industry are considered...". On this point I shall observe that perhaps the emphasis is due more to the desire to take the reader away from the traditional point of view — to which Sraffa refers just two paragraphs later — than to the unshakeable belief that any sort of changes in the produced quantities should be banned from the theory of value forever. In fact, we may easily understand how important it must have been for Sraffa that economists, who were used to thinking in terms of marginal changes, were forced to give up their habit and begin reasoning about value without any change at all. On the other hand, it is difficult to believe that Sraffa was unaware that in his Part III, even in the single production case, when a new method supersedes the method previously in use in a given industry, then, in general, a change in the scale of production in (potentially) all industries may be necessary in order to avoid the net product containing negative quantities for some commodities and positive quantities of uneatable means of production.[3]

Summing up, I think that a less rigid interpretation of Sraffa's statement is possible. Moreover, I think a more flexible approach is necessary to achieve a useful assessment of Sraffa's theory of joint production. A strict separation of the theory of value from that of changes in the scale of production has proved impossible in the case of joint production. Analysis of value necessitates, in some cases, the introduction of variations in the scale of production, even considering a given r, unless we want to take a purely axiomatic approach and admit negative prices. However, since Sraffa's theory does not aim at a simultaneous determination of prices, distributional variables, and produced quantities, then the theory of value may be seen as a first and separate step, while distribution, consumption and accumulation may be developed in subsequent stages. In the first step — theory of value — prices are analyzed under convenient simplifications on consumption and accumulation, in the second step consumption and accumulation should be fully introduced into the picture. Thus, for instance, in addition to constant returns to scale, the simplification adopted on the demand side by most of the works on joint production has consisted in taking a given and invariable consumption vector (vector y in system (PQ)), though not assuming — as Sraffa did — a given and invariable activity vector.

Let us now come to open problems. Taking for granted the above separation of the theory into subsequent steps, I think however that the

[3] For this observation, see I. STEEDMAN, "Returns to Scale and the Switch in Methods of Production", *Studi Economici*, 35, 1980.

results obtained in the theory of value are by now sufficient to deal resolutely with systematic analysis of distribution, consumption and accumulation. As regards theory, the results obtained in the case of a fixed y should be extended to the case where y is a function of r, p, q, A, B, l. Partial results, as far as I know, have been found by Salvadori and Schefold.[4] Note that the dependence of y on prices is not necessarily to be interpreted as adherence to the neoclassical standpoint. For instance, keep r and total employment as fixed, and suppose that workers and capitalists' consumption patterns consist of fixed baskets of commodities, but that those baskets are different across classes. Then demand would be different in different S_i's, due to different ratios of total wage to total profit.[5] Also the question as to whether the Cost-Minimizing Systems are square, i.e. contain as many methods as non-zero prices, should be re-examined in this light.[6]

On the empirical side, I think that explanations of the variations of aggregate consumption based on different behaviour of economic agents, due to their different social class and income, should be taken as a point of departure for data analysis and subsequent integration into the theory of prices. This may be seen as no more than a rhetorical incitement. And yet, if one considers that almost all contemporary empirical research is performed within the paradigm that actual economies work as if they contained a single intertemporally maximizing agent, then the importance of encouraging and supporting serious empirical research starting from the differences among agents, will be fully appreciated.[7]

4. The second open problem is the gravitation of market prices towards prices of production. I shall not even make an attempt at being exhaustive, nor shall I refer directly to classical economists. This would exceed by far the short paper I have been invited to write. Nonetheless, I think that an example and some considerations on recent work may stimulate discussion.

[4] N. SALVADORI, "Existence of Cost-Minimizing Systems within the Sraffa Framework", *op. cit.*, p. 287; B. SCHEFOLD, "The Dominant Technique in Joint Production Systems", *Cambridge Journal of Economics*, 12, 1988, pp. 117-118.

[5] See the works by Salvadori and Schefold just quoted.

[6] B. Schefold has shown that, apart from flukes, under the assumption of a fixed y the Cost-Minimizing Systems are square. SCHEFOLD himself, "The Dominant Technique in Joint Production Systems", *op. cit.*, p. 118, footnote 2, shows that when y is not fixed, then non-square Cost-Minimizing Systems may arise which are not flukes; but he argues that such systems would be unstable since they would not be able to match variations of demand outside the space spanned by the "few" methods they contain. This is a possible argument to rule out unpleasant non-square systems, but, as I shall try to show below, the stability-instability issue is very far from being solved either in single or joint-production systems.

[7] For an attempt to criticize the use of the representative agent in dynamic macroequations, see my "On the Dynamic Shape of Aggregated Error Correction Mechanisms", *Journal of Economics Dynamics and Control*, 12, 1988; and "On the Dynamics of Aggregate Macroequations: from Simple Microbehaviors to Complex macrorelationships", in *Technical Change and Economic Theory*, edited by G. DOSI, C. FREEMAN, R. NELSON, G. SILVERBERG, L. SOETE, London, 1988.

The example is a simplified version of a model put forward by Nikaido in 1983.[8] Basically, I have eliminated a feature that has been considered a departure from the classical setting of the problem[9] and yet obtain the same negative result. Now, in the case of single production, an escape route, which is endowed with a strong economic meaning, seems possible; but, as soon as joint production is allowed the same solution, does not generally lead to stability. Therefore, I argue, there is a specific stability problem with joint production. The model features are as follows:

(1) There are two industries in the model.

(2) The wage-rate is constant through the process and is given in terms of commodities.

(3) Capitalists move capital towards the industry in which the rate of profit is higher; the rate of profit they use in their decision at t is calculated using prices that have been observed at $t-1$.

(4) Prices at t move up or down, relative to prices at $t-1$, according to whether demand is higher or lower than supply.

(5) Supply at time t equals production decided at $t-1$ and carried out between $t-1$ and t.

(6) Production decisions at t (to be carried out between t and $t+1$) determine demand of means of production at t.

These assumptions may be easily formalized. Let A be the input matrix: a_{ij} is the quantity of commodity j that is necessary to produce one unit of commodity i; A includes the wage-commodities, therefore labour quantities l_i do not explicitly enter the price equations. To further simplify the matter I assume $r = 0$, so that capitalists do not consume or accumulate. Prices of production are:

$$p^* = Ap^*,$$

while equilibrium quantities are:

$$q^* = A'q^*$$
$$q_1^* l_1 + q^* l_2 = L,$$

where L is equilibrium employment (not necessarily full employment of course).

Now let p_{it} and q_{it} indicate market prices and actual quantities at t.

[8] H. Nikaido, "Marx on Competition", *Zeitschrift für Nationalökonomie*, 43, 1983.

[9] See, for instance, G. Dumenil and D. Levy, "The Dynamics of Competition: a Restoration of Classical Analysis", *Cambridge Journal of Economics*, 1987, pp. 141-143; P. Flaschel, W. Semmler, "Classical and Neoclassical Competitive Adjustment Processes", *The Manchester School*, 1987, p. 21.

Being interested only in local analysis around equilibrium I shall write all equations directly in linear form. Let

$$\tilde{q}_{it} = q_{it-1} + \alpha_i[p_{it-1} - (a_{i1}p_{1t-1} + a_{i2}p_{2t-1})] \tag{1}$$

be the the quantity of commodity i that capitalists *intend* to produce during period t, as a result of their moving to (from) industry i ("moving to" includes "remaining in" of course). Their decision is taken on the basis of profit calculated by means of prices at $t-1$; in our simple model, if one industry makes profits the other makes losses.

Let us now turn to market price formation:

$$p_{it} = p_{it-1} - \beta_i[q_{it-1} - (a_{1i}\tilde{q}_{1t} + a_{2i}\tilde{q}_{2t})]. \tag{2}$$

Thus, as expected, movements of market prices are determined according to the excess of supply, i.e. q_{it-1}, over the demand of commodity i that corresponds, via the matrix A, to intended productions $\tilde{q}_{1t}$ and $\tilde{q}_{2t}$. Capitalists will not in general realize their plans. Yet it is reasonable to assume that quantities actually produced follow the intended quantities, that is:

$$q_{it} = q_{it-1} + \gamma_i[p_{it-1} - (a_{i1}p_{1t-1} + a_{i2}p_{2t-1})]. \tag{3}$$

Coefficients α_i, β_i, γ_i all represent speeds of reaction. I assume all of them are small enough not to constitute, in themselves, problems for stability. Prices p^* and quantities q^* are a rest point for the dynamics defined by (1), (2), (3). In the Appendix it is formally shown that such a rest point is stable or unstable according to whether the first or the second of the following inequalities hold:

$$\frac{a_{11}}{a_{12}} \leq \frac{a_{21}}{a_{22}}$$

$$\frac{a_{11}}{a_{12}} > \frac{a_{21}}{a_{22}}$$

(if the first inequality holds strictly the model is asympotically stable). This is the result obtained by Nikaido for a model whose dynamics are specified in a more complex manner. Obviously, as there is no reason to assume that the first inequality holds in actual systems, the result must be considered as negative.

It is not easy to get an intuitive grasp of the above instability result. Yet a simple consideration may help. Capital movements in response to prices always go in the "right" direction. But, at the same time, such movements cause a change in demand. The latter depend on the coefficients of A and thus may go in the "wrong" direction. Conversely, the wrong presumption behind the idea that the classical model is unconditionally stable probably depends on concentrating attention on supply, neglecting the feedback of supply on demand (equation (2)).

It must be pointed out that in the model above markets need not clear. Therefore, as I do not introduce or explicitly model inventories, the resulting dynamics should be considered as no more than a preliminary exercise.[10] Yet I hypothesize that the introduction of inventories and other features necessary to improve the realism of the model would not cure the instability stemming from the shape of A. I hope future work will clarify this point.

Instead of seeking a solution to the above problem, one could be tempted by a different mechanism of price formation, i.e. a mark-up adjustment process. In its most naïve form, we may think of prices and quantities adjusting separately. As regards prices, assume that capitalists add their target mark-up to last period costs:

$$p_t = (1 + m)Ap_{t-1} + lw,$$

where now A does not contain wage-commodities, prices are measured in money, the wage rate is measured and *given* in money. This process is stable if m is smaller than the maximum rate of profit and, of course, converges to:

$$lw + (1 + m)Alw + (1 + m)^2A^2lw + \dots$$

I am aware that this is an extremely simplified representation of a mark-up process. Nevertheless, I think it contains the essential ingredients for my point. Now, let us turn to a joint-production system (B, A, l) and try to reproduce the mark-up process. First of all, given p_{t-1}, there exists only one vector of prices at t so that: (1) the price of each commodity is equal across all industries producing that commodity; (2) the above mark-up rule is reproduced (invertibility of B is needed):

$$Bp_t = (1 + m)Ap_{t-1} + lw.$$

This makes little economic sense, because pricing in each industry depends on the whole B. But even if this drawback is neglected, the process does not converge in general, for exactly the same reason that reduction to dated quantities of labour is generally impossible within joint production.[11]

Summing up, the classical adjustment process, as represented by price-taking agents driven by surplus profits, seems to be unable to produce general stability results, even in a single-product system. Mark-up pricing, on the

[10] A model containing joint dynamics of production and inventories is contained in Dumenil and Levy's paper quoted above. Yet, as the paper presents only the outcome of computer simulations, an assessment of the stability results claimed is impossible.

[11] As an extreme case, take $A = I$ and:

$$B = \begin{pmatrix} 2 & 2 \\ 2 & 2 + \epsilon \end{pmatrix}.$$

One of the eigenvalues of B is near to zero, so one of the eigenvalues of $B^{-1}A = B^{-1}$ is greater than 1.

contrary, is a stable process, but only insofar as single-production processes are concerned.

I conclude by noting that the theory of gravitation seems to be in an unsatisfactory state. It may be added that it has not been sufficiently studied as yet, perhaps because it has been considered a secondary issue. But this would be a serious misunderstanding. It may be true that market prices do not matter, but only as long as they behave well.

Appendix

To analyze the system (1), (2), (3) around equilibrium, i.e. around p^* and q^*, we first substitute (1) into (2), then rewrite (2) and (3) in matrix form:

$$-\begin{pmatrix} p_t \\ q_t \end{pmatrix} = (I+D)\begin{pmatrix} p_{t-1} \\ q_{t-1} \end{pmatrix}$$

$$D = \begin{pmatrix} <\alpha\beta> A'(I-A) & <\beta>(A'-I) \\ <\gamma>(I-A) & 0 \end{pmatrix},$$

where $<\alpha>$ is the 2×2 matrix having the α_i's on the main diagonal, and the same for $<\beta>$ and $<\gamma>$, while A' is the transpose of A. If D has an eigenvalue with positive real part then $I+D$ has an eigenvalue with real part greater than 1 and the process is unstable. As regards the eigenvalues of D, first note that the vectors:

$$\begin{pmatrix} p^* \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ q^* \end{pmatrix}$$

are eigenvectors corresponding to the eigenvalue 0. Therefore the characteristic equation of D must have zero as a root whose multiplicity is at least two. It will therefore have the form:

$$\lambda^4 + u\lambda^3 + v\lambda^2.$$

Computation of u and v gives (assuming $\alpha_i = \alpha$, $\beta_i = \beta$, $\gamma_i = \gamma$):

$$u = -\alpha\beta[a_{11}(1-a_{11}) + a_{22}(1-a_{22}) - 2a_{12}a_{21}]$$
$$v = \beta\gamma[a_{12}^2 + a_{21}^2 + (1-a_{11})^2 + (1-a_{22})^2].$$

Without loss of generality we may assume $1-a_{11} = a_{21}$, $1-a_{22} = a_{12}$. We have:

$$u = -\alpha\beta\left[\frac{a_{11}}{a_{12}} + \frac{a_{22}}{a_{21}} - 2\right] a_{12}a_{21}.$$

But if $a_{11} > a_{12}$, then $1-a_{11} = a_{21} < 1-a_{12} = a_{22}$, and so on. Thus $u < 0$ if and only if:

$$\frac{a_{11}}{a_{12}} > \frac{a_{21}}{a_{22}}.$$

The non-zero eigenvalues are:

$$\frac{-u \pm \sqrt{u^2-4v}}{2}.$$

It is easily seen that one of them has positive real part if and only if $u < 0$.

Putting $\alpha_i = 0$ the above example may be considered as the discrete-time linear counterpart of the model recently put forward by Flaschel and Semmler.[12] In fact, the equation of price change in Flaschel and Semmler is the following (with small adaptations and changes in notation):

$$\dot{p} = - < d > \ < p > (I - R^* A')x,$$

where $< d >$ is a diagonal matrix of speeds of reaction coefficients, while $< p >$ is the matrix with the prices on the diagonal (therefore the equation is in the rate of change of prices).[13]

Apart from the continuous-time representation, rate-of-change form and non-zero rate of profit, such an equation works like my equation (2) (after elimination of the $\tilde{q}_i(t)$'s by means of (1)) with $\alpha_i = 0$. In other terms, in their model any influence of $\dot{x}$ on $\dot{p}$ is excluded. By consequence, the matrix corresponding to my D has the form:

$$\begin{pmatrix} 0 & C' \\ -C & 0 \end{pmatrix}$$

and the model is stable. Had they allowed — as Nikaido does — for influence of $\dot{x}$ on $\dot{p}$, dependence of stability on the shape of A would not have been escaped. But, on the other hand, the exclusion of any influence of $\dot{x}$ on $\dot{p}$ does not seem economically sensible, as is apparent from the discrete-time version of the model.

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[12] "Classical and Neoclassical Competitive Adjustment Processes", *op. cit.*

[13] P. Flaschel and W. Semmler, "Classical and Neoclassical...", *op. cit.*, p. 23. The same observation made above on the issue of market clearing and inventories holds for the Flaschel and Semmler model; see p. 24, footnote.