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Cyclical Growth and Long-Term Prospects*

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Introduction

In the standard trade-cycle theories that follow the "Keynesian" tradition, the main part of investment behaviour is explained exclusively by current conditions. The projection of those conditions into the future, when coupled with sufficiently high multiplier and accelerator parameters, leads to the dynamic instability of the long-term path of the economy and, in the presence of floors and ceilings and/or non-linearities, gives rise to regular, endogenous cycles.

The present paper suggests how the dynamic properties of a well representative model in the above class would be modified by the influence on investment of slow-changing long-term prospects (relatively independent of current conditions). We suggest that these prospects affect the sensitivity of investment to current conditions.

1. Investment decisions in the real models of the business cycle

The term "real models of the business cycle" is used here to indicate the dynamic theories of output anticipated by M. Kalecki's *Essay on the Theory of the Business Cycle* published in Polish in 1933; these theories abounded after the publication of Keynes's *General Theory* up to the mid-fifties. The coupling of some version of the consumption multiplier with the idea that investment expenditure is positively related to the current activity level and/or growth rate plays an essential role in all such theories;

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age and simplicity notwithstanding, they are still in focus for modern economists engaged in macrodynamic theory along Keynesian lines.

In the present paper this group of models will be examined from a particular viewpoint. We shall overlook the sharp and obvious differences existing among them and concentrate on a single common feature relating to their attitude towards investment decisions and the consequences of this attitude. This is not to say, of course, that the models concerned have similar investment functions: for example, some of them, such as Harrod's and Hicks's, incorporate the acceleration principle, while others do not.

The common feature stressed here is that investment activity is divided into two parts, which are governed by two different sets of expectations. One set is relatively independent of current activity levels and is either considered as exogenous, or as a slowly changing function of time, which may be influenced by the past history of the system. The other set of expectations is exclusively related to currently observed results concerning profits and/or output, given available capacity.

The above dichotomy, wich is reflected in the distinction between autonomous and induced investment, is open to objections, some of which were clearly recognized at the time when the first conlcusions from the fast growing literature on the real trade cycle models were being drawn.[1]

These critical remarks aimed more at qualifying the notion of autonomous investment than at casting doubts on the adequacy of the twin-notion of induced investment; still, important doubts can be raised, and *were* raised, e. g. by Keynes,[2] on the ground that the high sensitivity of large components of investment expenditure to current results is exaggerated to the point of explaining such expeditures *exclusively* in terms of these results. If we confine our attention to output expectations (which are most relevant in the case of induced investment[3]) we may observe that, although necessary, favourable expectations regarding the immediate future (as determined by current results) are not sufficient to explain large flows of investment. They are necessary, for otherwise even long-lived projects would be postponed; but they are not sufficient, as long as the useful life of a large share of capital goods cannot be confined without loss to the immediate future.[4]

[1] HARROD, in particular, was inclined to stress that the distinction was more a matter of degree than of kind, in the sense that investment expenditures, that could be safely deemed as independent of current income, would not presumably be unaffected by a long depression. He insisted that though "in a sufficiently short period all investment is autonomous", no investment expenditure is independent of output when the latter is measured over a sufficiently long time Interval. R. Harrod, "Notes on Trade Cycle Theory", *The Economic Journal*, June 1951, p. 267.

[2] J. M. KEYNES, *The Collected Writings of J. M. Keynes*, edited by D. E. Moggridge and E. Johnson, London, Macmillan, 1971 ff., vol. XIV, pp. 152-3.

[3] In so doing, we abstract from price expectations (including the prices of inputs) and from technological expectations (to be considered among the determinants of autonomous investment).

[4] Harrod's remarks on this point are conclusive only in as far as necessity is concerned. "*You think I am wrong in making investment a function of current growth only. Granted. Suppose*

It should be noticed that, if we introduce long-term expectations into the analysis, the usual notion of autonomous investment must be also modified. The existence of components of investment expenditure (typically, innovation expenditure), bearing little or no relation to current levels of activity and rates of capacity utilization, by no means implies that such expenditures bear no relation to output expectations in general.

For the above reasons, the definitions of autonomous and induced investment adopted in the present paper are more general than the usual ones. The label "autonomous investment" is used here to indicate a flow of expenditure which is independent of current rates of capacity utilization; by contrast, induced investment is responsive to current (output) results.

2. Endogenous cyclical growth with given long-term expectations

2.1. In what follows we assume that long-term expectations are exogenously given. Our aim here is to emphasize the following related points:

a) The sensitivity of investment decisions to current results cannot be convincingly specified without referring to a state of long-term expectations.

b) It follows from *a*) that long-term expectations have a definite influence on cyclical behaviour and long-run economic performance; more generally, the existence of persistent cycles is conditional on particular states of long-term expectations.

Statement *b*) is proven here only in relation to a particular dynamic model, which borrows most of its basic formal structure from Kaldor's 1940 trade cycle model.[5] Thus, the analysis to follow retains the simple aggregative

only half were governed by current growth, the rest by long period planning. My theory is substantially intact. It remains true that the growth of consumption cannot slow down without producing a great recession; but in this case the recession would only have to be such as to reduce savings to half their usual level. Personally I believe by far the greater part of investment rests on an immediate prospect of an increase of demand. People do not build new factories for use some years hence nor houses that will remain unwanted. Why should they? They increase equipment at the last feasible moment to save interest. Moreover if you try looking more than a year or so ahead everything becomes so violently uncertain." R. Harrod, letter to J. M. Keynes, in J. M. Keynes, *The Collected*, *op. cit.*, pp. 175-6. This passage was brought to my attention by an unpublished paper of Prof. Hainz Kurz. One may observe, in passing, that Harrod's argument rests on the implicit assumption that autonomous investment has a sufficient degree of persistence over time. By "sufficient degree of persistence" we mean that the observed volatility in the flow of autonomous investment expenditure, must be consistent with the idea that current realized results can be taken as a reliable guide to the near future. If this were not the case, expectations, formulated on the basis of current results, would be systematically disappointed by the effects of volatile autonomous ependiture on demand levels over the near future.

[5] N. Kaldor, "A Model of the Trade Cycle", *The Economic Journal*, vol. L, 1940, pp. 78-92. The basic mathematical structure of Kaldor's model has been thoroughly investigated in the literature; cf., in particular, S. Ichimura, "Toward a General Nonlinear Macrodynamic Theory of Economic Fluctuations", in K. K. Kurihara (ed.), *Post-Keynesian Economics*, London, Allen & Unwin, 1954; W. W. Chang and D. J. Smyth, "The Existence of Cycles in a Non-linear Model: Kaldor's 1940 Model Reexamined", *Review of Economic Studies*, vol. 38, 1971, pp. 37-44;

structure of the standard real models of the business cycle. This is a useful (if entirely preliminary) step, in so far as it makes the dynamic implications of the general argument raised far more intuitive.[6]

The following section deals with proposition *a)*; section 2.3 presents the formal model, which exhibits properties *b)*; these properties and the methodological underpinnings of the model are informally discussed in the final section of part 2. The appendix contains an outline of the main mathematical proofs.

2.2. The shape of the investment function proposed by Kaldor in his 1940 trade-cycle model was later interpreted by him in terms of volatility of expectations triggered by current results.[7] A cyclical endogenous dynamics for overall capital stock and output is obtained by Kaldor in this model. Similarly, investment behaviour (and cyclical instability) in the other endogenous business-cycle models in the Keynesian tradition is strictly related to the projection of current results concerning output into the future.[8] In the present section we lay the premises for the construction of an investment function which is formally akin to that proposed by Kaldor in 1940, but where static or extrapolative expectations play no role and expectations concerning long-term growth are given.

The influence of current results on a firm's investment decisions must depend on the extent to which they are allowed to influence the path of the expected returns associated with (some or all of) the investment projects available to the firm. Since most economic activities are time-consuming and are characterized by important factors of continuity, the observed rate of change of economic variables tends to be constrained within relatively narrow bounds. Thus, it is generally agreed that the current states of capacity and output are likely to affect investment decisions, in as far as they are bound to affect at least the early expected returns of some relevant investment project. This implies that the current state of output must affect investment decisions even if entrepreneurs do not take the present as a guide to the distant future.

To make this simple idea more precise, let us assume a world of rigid prices, where monetary policy intervention holds the rate of interest constant.

Let us define "normal output", at any given date t, the trend level of

H. R. VARIAN, "Catastrophe Theory and the Business Cycle", *Economic Inquiry*, 1979, pp. 14-28; F. CUGNO and L. MONTRUCCHIO, "Stabilty and Instability in a Two Dimensional Dynamical System: A Mathematical Approach to Kaldor's Theory of the Trade Cycle", in G. P. SZEGO (ed.), *New Quantitative Techniques for Economic Analysis*, New York, Academic Press, 1982.

[6] Implications in more complex (and realistic) settings will be analyzed in future work.

[7] N. KALDOR, "The Relation of Economic Growth and Cyclical Fluctuations", *The Economic Journal*, vol. LXIV, n. 253, pp. 53-71.

[8] Hicks is most explicit on this point; see J. R. HICKS, *A Contribution to the Theory of the Trade Cycle*, Oxford, Clarendon Press, 1950, p. 39 and p. 43.

output at t. It is assumed that entrepreneurs hold prior conjectures on the expected trend of output over the planning horizon $(0, \infty)$. These conjectures are relatively independent of current conditions, in the short-run; more precisely, it is assumed that the sensitivity of the expected normal-output levels, to changes in current output, is sufficiently small. Thus, expectations concerning normal output can be taken as given exogenously, at least in a short-run[9] analysis.[10]

Of course, current output may well have a large influence on expectations concerning output, rather than normal output; in particular (since the expected output path is continuous), a higher (lower) current output implies a higher (lower) expected output over the initial part of the planning horizon. In the present paper this part is defined "immediate future". Output expectations referring to the whole planning horizon (output which one expects to be willing to produce at each date in the future in the absence of capacity constraints) are defined "long-term (output) expectations".[11]

For simplicity, long-term expectations are understood to be uniquely determined by current output and by prior conjectures concerning the trend, synthesized by the (given) expected average growth rate.

Since prices are held constant throughout, a state of long-term output expectations, with given current levels of a firm's output and capacity, is assumed to be in one-to-one relation with a state of expectations concerning the present value of each investment project available to the firm. Obviously enough, as long as the rate of interest is positive, a higher current output gives rise to higher present value expectations in so far as returns expected over the immediate future would improve; the effect of a larger capacity goes the other way, since a bigger share of potential sales could then be met by the existing capacity. Still, the crucial claim has not yet been justified, namely that investment decisions may be highly responsive to current conditions, even if these are not allowed to alter expectations concerning normal output. If this responsiveness were to rest only on the influence of current output on the expected yields (generated by available investment projects) over the immediate future, the claim would be quite arbitrary,

[9] Here the term "short run" does not refer to the Marshallian short run, although, like the latter, it can not be identified with a definite interval of calendar time. It is defined only in terms of the relative reaction speed of the variables involved.

[10] Conditions under which the separation of the short-run from the long-run in dynamic models does not lead to serious mistakes can be rigorously expressed. Cf. H. A. SIMON and A. ANDO, "Aggregation of Variables in Dynamic Systems", *Econometrica*, vol. 29, April 1961; L. BOGGIO, "The Stability of Production Prices in a Model of General Interdependence", in W. SEMMLER (ed.), *Competition, Instability and Nonlinear Cycles*, Berlin, Springer Verlag, 1986; H. Haken, *Synergetics*, Berlin, Springer Verlag, 1978.

[11] Our definition contrasts with Keynes's notion of a state of long-term expectation, which refers to the psychological expectation about the determinants of the prospective yields over the whole life of an investment project.

for the pay-back limit[12] of long-lived investment projects seems to lie well beyond the immediate future.

The above claim bears on the postponement of investment projects, which are otherwise profitable (given long-term prospects), when the returns they are expected to yield over the immediate future are low, because existing capacity is higher than is required to meet expected demand over this interval. In this way, sufficiently optimistic long-term conjectures and a sufficiently high current output are both necessary conditions for the current implementation of long-lived investment projects. The remark has a further crucial implication. The higher the optimism in long-term conjectures, the larger the size of long-lived investment decisions at a given and sufficiently high level of output; but, *ceteris paribus*, such investments would not be undertaken with a sufficiently low capacity utilization; thus, the implication is that the sensitivity of long-lived investment decisions to changes in current output depends on the optimism of long-term conjectures.

2.3. The demand for investment goods comes from investment decisions A, which are independent of current output, and from decisions induced by the state of output and capacity. As we shall see, both decisions also depend upon growth prospects π. Thus we have,

$$I = I(Y, K, \pi) + A, \quad \text{with} \quad I_Y > 0,\ I_K < 0,\ I_\pi > 0. \tag{1}$$ [13]

As is well known, the influence of activity levels and of capital stocks on the demand for investment is interpreted by Kaldor in terms of the incentive to invest determined by current profits. The non-linearity of his investment function $I = I(X, K)$, where activity levels are measured in terms of employment (X) rather than income, is explained as follows. When activity levels are low with respect to the capital stock, a rise in activity and hence in profits has a small impact on investment because of undesired excess capacity. The same is true when activity levels are very high, since increasing construction and borriwing costs imply that profits do not rise in line with activity.[14]

[12] This is the date at which the undiscounted cumulative expected yields of a project equals its cost.

[13] I_Y, I_K, I_π indicate the first partial derivatives of I( ) with respect to Y, K and π, respectively. Second partial derivatives of I( ) are expressed as follows (with obvious notation): I_{YY}, I_{YK}, $I_{Y\pi}$... The same notation for partial derivatives applies to all functions of a vector variable appearing in the paper, and it is occasionally extended to indicate derivatives of functions of a scalar variable.

[14] The increasing difficulty met by entrepreneurs in obtaining credit is also mentioned by Kaldor in this context; this, however, seems to have more to do with realized investment than with desired (ex-ante) investment. The rising construction costs referred to in the text are to be explained by the declining marginal productivity of labour in the face of shorth-run capacity

Turning now to our measure of activity levels in terms of output, it is easy to see how an alternative interpretation of Kaldor's investment function can be given in terms of demand rather than profits, which amounts to adopting a non-linear stock-adjustment principle. Although we assumed that prices, wages and the interest rate were all constant, we can avoid placing great emphasis on smooth substitutability (between capital and labour) in order to find solid grounds for the declining influence of output on investment at high activity levels (for a given capital stock). This is easily achieved if we consider the influence on investment decisions of non-linear borrowing costs, associated with Kalecki's principle of increasing risk.[15]

Aside from these amendments and additions, investment function (1) differs from Kaldor's in one essential respect: the existence of one more independent variable, namely, π. As long as we hold π fixed, we can formalize the above properties assuming that I( ) is twice continuously differentiable and that the partial derivative I_Y first increases and then decreases in Y, for a given K. Things get more complicated if we allow for parametric changes in π, since our main point is that I_Y depends on π; but, again, this relationship is non-linear, as is argued below. Let us consider how the second partial derivative $I_{Y\pi}$ changes with parametric increases of π at given levels of Y and K (see fig. 1). If the given Y is sufficiently low, $I_{Y\pi} > 0$ no matter how high the level of π is. The idea here is that with very unfavourable current results, the current rate of accumulation is always below the level at which the upper constraints imposed by capacity and increasing risk[16] start to bite. This is related to the postponement of investment projects alluded to in the previous section. At higher levels of Y we still have, initially (i. e. for low π), $I_{Y\pi} > 0$, but for a sufficiently large π, investment demand is eventually pushed up into the region where the above-mentioned upper constraints increasingly bite, so that $I_{Y\pi} < 0$. We now observe that the grounds for the upper non-linearity of the function $I(Y, K, \pi)$, for given K and π, are quite different, depending on the level of π. For a sufficiently high π the sensitivity of

constraints. The interpretation of the upper non linearity of Kaldor's investment function in terms of construction costs seems to require one of the following assumptions (or a mix of them). Smooth substitutability between capital and labour with constant commodity prices and wages; capacity constraints are more stringent in the capital-goods industries, and a smooth relative increase in the price of capital goods occurs as activity levels keep rising beyond certain levels (capacity constraints must here curb desired rather than realized investment).

The stress on increasing construction costs raises a further complication within Kaldor's model; labour productivity and profits must increase (at given prices), as the capital stock gradually increases when the system is in short-run stable effective-demand equilibrium at high levels of activity. Kaldor's implicit assumption is, that the above positive effect of accumulation on investment is not sufficient to counteract the negative effect, due to productive capacity rising faster than output and profits.

[15] M. Kalecki, "The Principle of Increasing Risk", *Economica*, 1937.

[16] We avoid reference to changes in the rate of interest, on the assumption of an accomodating monetary policy.

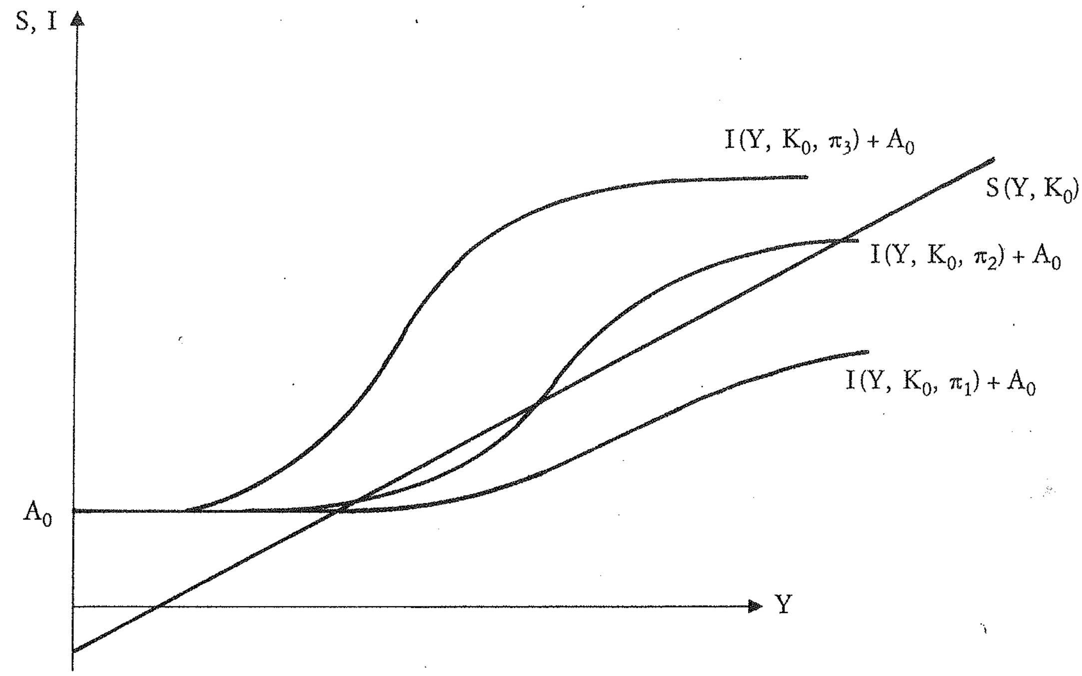

Fig. 1

Graphs of the savings function $S(Y, K_0)$ and of the investment functions $I = I(Y, K_0, \pi_j) + A_0$, $j = 1, 2, 3$; $\pi_1 < \pi_2 < \pi_3$. K_0, A_0 and each alternative π_j denote given and costant values of K, A and π, respectively.

investment demand to increases in output (i. e. I_Y) is eventually curbed, as Y increases, only by borrowing and construction costs. For lower levels of π (i. e. for less optimistic long-term expectations), instead, I_Y is eventually curbed by the increasing divergence between the current rate of accumulation I/K and the expected rate of long-term growth π. This constraint acts, at such lower levels of π, before the other upper constraints (borrowing and construction costs) start to bite; the lower π, the lower the level of Y at which the divergence between current growth and expected long-term growth is felt in its effects on I.

As suggested above, our assumptions make it possible to carry out the analysis of the short-run dynamics for an exogenously given level of π; thus, in what follows, π is dealt with as a parameter. In this case, expression (1) boils down to

$$(1.b) \qquad I = {}_{\pi}I(Y, K) + A, \quad \text{with} \quad {}_{\pi}I_Y > 0, \quad {}_{\pi}I_K < 0;$$

to simplify our notation we shall often omit parameter π, unless it is strictly necessary.

From the view-point of economic interpretation, the main difference between (1.b) and Kaldor's investment function is as follows. While the demand for investment is postively affected by the level of output at a given capacity, the upper non-linearity of this relationship is not necessarily explained by increasing borrowing and construction costs. A further cause for the non-linearity of the investment function is that entrepreneurs must be reluctant to undertake rates of accumulation I/K that happen to be more and more at variance with the expected long-term growth π.[17]

At the given and constant price structure, aggregate output Y results from the production of investment goods I_0 and from the production of consuption goods C_0; this yields the definition $Y \equiv C_0 + I_0$.

By following a clear-cut Keynesian view of the relationship between savings and investment, we assume here that investment decisions are always realized, or as Kaldor puts it, that ex post saving is adjusted to ex ante investment; this is formalized as

(2) $$I_0 = I.$$

Assuming a constant rate of exponential depreciation δ we obtain

(3) $$dK/dt = I - \delta K.$$

Kaldor's savings function is now introduced, where savings depend positively on income levels, and negatively upon the level of capital stock.[18]

(4) $$S = S(Y, K) \quad S_Y > 0 \quad S_K < 0.$$

Desired savings are always strictly lower than income and increase with the latter without (upper) bounds; taking also into account the upper non-linearity of the investment function $I(Y, K, \pi)$, for given K and π, we are led to formulate the following plausible Kaldorian assumption.

(5) Whatever the level of autonomous investment, savings are lower (higher) than investment if gross-output is sufficiently small[19] (large).

If, following Cugno and Montrucchio,[20] we assume that I(Y, K) and S(Y, K) are homogeneous in the first degree in Y and K, using (1) and (4) we obtain $I/K = {}_{\pi}I(Y/K, 1) + A/K$, $S/K = S(Y/K, 1)$. For simplicity of notation, we define $C_0/K \equiv x$, $A/K \equiv z$, $Y/K \equiv y$, $I(Y/K, 1) \equiv I(y)$.

In the framework of the present paper, the second Kaldorian assumption on the savings and investment functions holds true if, and only if, growth expectations are sufficiently optimistic. Mathematically the assumption is expressed as follows.

(6) For each $\pi > \pi^*$ there exist unique real numbers y_1, y_2, with $0 < y_1 < y_2$ such that ${}_{\pi}I_y = S_y$ at $y = y_1$ and $y = y_2$. Moreover $I'_y < S'_y$ at y, if $0 < y < y_1$ or $y > y_2$, while the opposite holds true if $y_1 < y < y_2$.

[17] See above, section 2.2.
[18] The idea is that real wealth has a positive influence on consumption.
[19] Recall that, by definition, gross output cannot be lower that autonomous investment.
[20] Cf. F. Cugno and L. Montrucchio, "Stability..." *op. cit.*

A diagrammatic representation of (6) is given in fig. 1, where $\pi_1 < \pi^* < \pi_2 < \pi_3$. For given K and π the above assumption defines an intermediate region of activity levels (Y), where the (marginal) propensity to invest is higher than the (marginal) propensity to save. If savings equal investment at a point in this region, this equality can be preserved, after a small increase in activity, only through a simultaneous fall in autonomous investment. To ensure that this fall will suffice to do the job, we assume the following.

(7) Let y_1 and y_2 be the output/capital ratios defined in (6) for a given $\pi > \pi^*$. Then there exists $z_1 > 0$ sufficiently small s. t. $S(y) > I(y) + z$ for $y_1 \leq y \leq y_2$, if $0 \leq z \leq z_1$.

Although the responsiveness of induced investment to current demand depends on long-term expectations, an adequate level of current demand is still a necessary condition to obtain positive induced investment. Thus, whenever activity levels in the consumer-goods industries are infinitely small with respect to the overall capital stock, the existence of a positive flow of investment is conditional upon the existence of a positive flow of autonomous investment. This is formalized as

(8) If $x = 0$, then $I(y) + z > 0$ if and only if $z > 0$.

Using the definition of Y together with (2), and recalling that $I'_y < 1$, we can define the investment/capital ratio as a function of x and z:[21]

(9) $$I/K = {}_\pi i(x, z) \quad {}_\pi i_x > 0 \quad {}_\pi i_z > 0.$$

Likewise, also taking into account (9),

(10) $$S/K = S({}_\pi i(x, z) + x, 1),$$

wich allows us to define the function ${}_\pi s(\ )$ as:

(11) $$S/K = {}_\pi s(x, z) \quad {}_\pi s_x > 0 \quad {}_\pi s_z > 0.$$

When the demand for consumer goods $C = Y - S$ differs from the production C_0, consumers' claims are disappointed; this drives producers to change the degree of capacity utilization, since they do not adjust their capital stock as fast as they adjust their output. Indeed, in accordance with

[21] Using the definition of Y together with (2), we can define the functions $g\colon R_+ \rightarrow R_+$ and $h\colon R^2_+ \rightarrow R_+$ such that $g(y) = y - I(y) = x + z = h(x, z)$. Since $I_y < 1$ the function g^{-1} exists, and the composite function $g^{-1}h$ is such that $g^{-1}h(x, z) = y$. The induced-investment function $c\colon R^2_+ \rightarrow R_+$ is such that $c(x, z) = g^{-1}h(x, z) - h(x, z) = I(x + z + c(x, z))$. It follows that $c_x = c_z$ on R^2_{++}. We can eventually define: $I/K = {}_\pi i(x, z) = {}_\pi c(x, z) + z$, where ${}_\pi i_x > 0$, ${}_\pi i_z > 0$. It is easy to see that $i_x = c_x$ and $i_z = c_z + 1 = i_x + 1$.

the Marshallian distinction between short run and long run, changes in the output of consumer goods, induced by demand, take place faster than the capital stock is allowed to change, as a result of current investment. Thus C_0/K increases, remains constant, or decreases over time, depending on whether C is larger than, equal to, or lower than C_0. Following Cugno and Montrucchio,[22] this is formalized as:

$$(12) \qquad (dC_0/dt)/C_0 - (dK/dt)/K = \mu(C/K - C_0/K),$$

where μ is a positive parameter.

So far nothing has been said about autonomous investment A. By definition, the level of A is not decided with reference to the current state of capacity utilization. Nevertheless it would be highly unrealistic to assume that autonomous expenditures do not bear any relation to the size of the economy, as emphasized by M. Kalecki and by J. Steindl after him.[23] Consider for example how the decision regarding plant capacity for the production of some new information technology would be influenced by the number and size of its potential users. This suggests that the timing of autonomous investment is not totally independent of cyclical considerations concerning the size of capital stock. In a model like ours, where production decisions are determined by demand only, but where investment decisions also depend on prior conjectures about long-term prospects, there seems to be good reason for entrepreneurs to believe that fluctuations in capital stock will be less pronounced than fluctuations in output; thus, the influence of K on autonomous expenditure A has some rational foundation within our model. Still, autonomous investment is partly exogenous, in so far as it depends on technological and/or institutional factors. For a constant rate of technical progress there is a target ratio A/K which entrepreneurs want, on the average, to attain and which depends, *inter alia*, on the proportional destruction of capital caused by the prevailing forms and speed of technical change.

Autonomous expediture is also influenced by long-term output expectations. To see why, suppose that the actual ratio A/K coincides, in the given initial conditions, with the target ratio A/K, while the expected long-term growth rate is π. Consistence between entrepreneurs' aims and expectations requires that they increase autonomous investment at the (proportional) rate π.

Thus, autonomous expenditure is ruled by three factors: (i) the target ratio $A/K = z^*$, as determined in particular by technological expectations;[24]

[22] Cf. F. Cugno and L. Montrucchio, "Stability..." *op. cit.*, p. 267.

[23] Cf. M. Kalecki, "Trend and Business Cycles Reconsidered", *The Economic Journal*, vol. LXXVIII, n. 310, June 1968, pp. 263-76; J. Steindl, "Ideas and Concepts of Long Run Growth", *Banca Nazionale del Lavoro Quarterly Review*, vol. 34, March 1981, pp. 35-47.

[24] It should be noticed, in passing, that technological expectations may also help to shape long-term expectations as defined above.

(ii) long-term (output) expectations; (iii) the gap between z and z* and the adjustment rule followed by entrepreneurs. $(dA/dt)/A$ is pushed further and further above (below) π, when z gets increasingly lower (higher) than z*. When the two ratios are equal, the proportional rate of change in autonomous expenditure will then reflect the state of long-term expectations[25] (fig. 2). The above assumption is formalized as

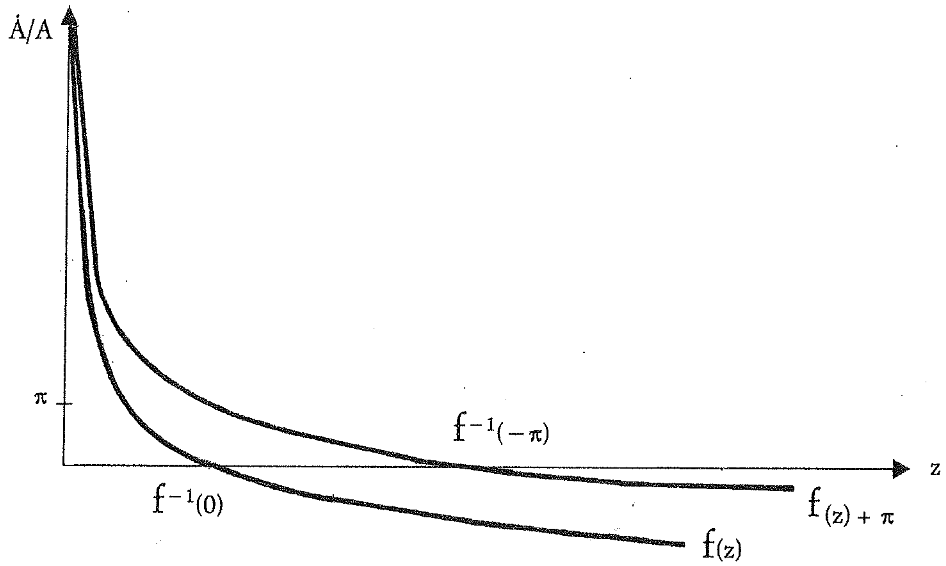

Fig. 2

(13) $(dA/dt)/A = f(z) + \pi$, with $f_z < 0$; there exist $z > 0$ s.t. $f(z) = 0$.

In the above expression the target ratio z* is implicitly defined $f^{-1}(0)$ and is constrained to be strictly positive. It is important to observe that, if the system state is not too far from equilibrium, the absolute value of the first derivative f_z should be expected to be relatively small, for the simple reason that autonomous investment is related to the size of the economy only in a long-term sense. Indeed, for z greater (lower) than z* and thus $(dA/dt)/A < \pi$ ($> \pi$), any further increase (decrease) of z would be interpreted in the sense that K has temporarily departed from its expected long-term behaviour. Adjustments in the rate of growth of autonomous expenditure would, as a consequence, be relatively slow. This restriction imposed on the adjustment speed of $(dA/dt)/A$ is less significant when the flow of A becomes infinitely small. With a zero initial level of autonomous expenditure, any positive finite time rate of change A, however small, would

[25] Although starting from Kalecki's idea that autonomous expenditure is influenced by the size of the economy, the above picture departs in some respects from Kalecki's view of the matter. Cf. M. Kalecki, "The Trend...", *op. cit.*

lead to an infinite growth rate $(dA/dt)/A$. These remarks suggest that the relationship between $(dA/dt)/A$ and z is strongly non-linear for z close to zero. In particular, we are induced to formulate the following assumption:

$$\lim_{z \to 0} f(z) = \infty. \tag{14}$$

Reflecting the fact that the rate of growth of autonomous expenditure also depends on long-term expectations, $(dA/dt)/A = \pi$ if $z = z^*$, and the level of A increases through time if $A/K < f^{-1}(-\pi)$. $f^{-1}(-\pi)$ indicates the value of the ratio A/K such that autonomous expenditure is constant through time if the expected long-term growth is π. It is worth observing that if $\delta < f^{-1}(-\pi)$ a ratio $A/K \leq \delta$ cannot persist over time; the existence of an upward drift in the long-term growth path would already be explained by circumstances exclusively related to autonomous expenditure. This is not generally the case, if $\delta > f^{-1}(-\pi)$.

After simple algebraic manipulations we obtain the following differential system:

$$(15) \qquad \begin{aligned} (dx/dt)/x &= \mu[{}_{\pi}i(x, z) - {}_{\pi}s(x, z)] \\ (dz/dt)/z &= f(z) - {}_{\pi}i(x, z) + (\delta + \pi) \end{aligned}$$

2.4. The formal analysis of the above model is left to the mathematical appendix; here, in a very intuitive way, we present its main properties, with the aim of discussing some further issues of interpretation and relevance.

a) On the assumption that system (15) has a unique and unstable steady-state position (x_e, z_e), the endogenous variables x and z follow a path moving around (x_e, z_e), and converge to a so-called limit cycle. Akin to Kaldor's model, the instability of (x_e, z_e) and hence the existence of persistent cyclical behaviour (limit cycles) requires that, in steady state, the desired investment ratio I/K responds to a change in the output of consumer goods, and hence in C_0/K, faster than the desired savings ratio S/K. In mathematical terms we must have $i_x - s_x > 0$ at (x_e, z_e).[26]

For a given δ, the necessary condition $i_x > s_x$ is satisfied only if the value of π falls in an "appropriate" bounded interval;[27] indeed, in this

[26] In the present model this condition is necessary, but not sufficient for the instability of (x_e, z_e), unless one assumes an infinite adjustment speed of consumer-goods output (to demand). The necessary and sufficient condition requires that $i_x - s_x$ is sufficiently larger than zero at (x_e, z_e). For given δ, the qualitative description of the set of parameter vectors (μ, π) meeting (26) may change, depending on further (and quite arbitrary) restrictions on the behavioural functions. A particular representation is given in apprendix B.

[27] In the context of a Kaldor model with exogenous growth rate, persistent cycles occur only if this rate falls in an appropriate interval (more generally, in an appropriate set of intervals). Cf. F. Cugno and L. Montrucchio, *op. cit.*, pp. 276-7.

respect, the influence of π is twofold, since π affects both the sensitivity of induced investment to current results and the rate of growth of autonomous expenditure. Appendix B shows how the joint action of the two influences is such that the steady state growth rate $g = i(x_e, z_e) - \delta$, the steady state consumption ratio x_e and income ratio y_e increase with π (for a given δ),[28] provided $i_x \geq s_x$.

If, starting from an "appropriate" value (meeting the condition $i_x - s_x > 0$), π keeps *increasing*, the difference $({}_\pi i_x - {}_\pi s_x)$ at each point of a region in state space must also increase;[29] still, the influence of the shift in steady state position must eventually prevail and make the dynamic equilibrium stable. The reason is that at points (x_e, z_e) such that $y_e = i(x_e, z_e) + x_e$ is sufficiently high, i_x is constrained by borrowing and construction costs. Likewise, if π keeps *decreasing*, starting from the same "appropriate" value, the steady state equilibrium eventually becomes stable.[30]

The above features of the model suggest how the emergence of endogenous cycles is related to the state of long-term expectations.

b) It has already been stressed how the present paper investigates only the short-term dynamics generated by prior conjectures; we considered explicitly, among them, only those which refer to the long-term rate of growth. The assumptions of the paper do not make sure that all conjectures are fulfilled (in particular, that $g = \pi$). Within this framework, the persistence of endogenous disequilibrium cycles requires that revised conjectures generate a new short-term dynamics of an unchanged qualitative nature.

In our interpretation, the adjustment rules in the model are chosen by entrepreneurs on the base of a conjecture-formation rule such that the expected path of output $Y^e(t)$ is a function of current conditions and of a common belief concerning the rate of growth π. Again, this function involves conjectures about the prevailing adjustment rules. The crucial point is that conditions can be specified such that actual adjustment rules are generated by conjectured adjustment rules, which share the same qualitative characteristics.[31] This implies that the qualitative features of economic

[28] See below, p. 24.

[29] To see this, first notice that the difference referred to above is a strictly increasing function of the difference ${}_\pi I_y - S_y$ (see below, p. 20). While S_y does not depend on π (we neglect the influence of long-term expectations on consumers' behaviour), ${}_\pi I_y$ (always increases with π if π is sufficiently low, or in any case if y is not too high (in which case investment behaviour is not yet constrained by borrowing and construction costs).

[30] For a proof of the above propositions, see appendix B. Notice that, if long-term expectations are sufficiently pessimistic, and hence π is sufficiently low, $(i_x - s_x) < 0$ everywhere in state space; persistent trade cycles cannot occur, no matter what is the size of parameters other than π.

[31] On this issue cf. M. Caminati, "Cicli endogeni in disequilibrio e congetture", *Quaderni del Dipartimento di Economia Politica*, Università degli Studi di Siena, Siena, 1990.

behaviour, which are responsable for endogenous disequilibrium business cycles, would not be destroyed by a growing awareness concerning macroeconomic behaviour.[32]

To follow this line of reasoning, one is however required to show why consumers and entrepreneurs do not have an incentive to modify their decision rules, after they realize how undesirable aggregate outcomes are related to such rules. Here we are confronted with a lack-of-coordination problem. With reference to the present model, entrepreneurs and consumers may collectively act, ex hypothesis, so as to stabilize the steady state position. For example, if all entrepreneurs agreed to make long-term investment less vulnerable to short-term depressions, then such depressions would not come about, or would be milder. But the isolated entrepreneur who acted in the same sense would only suffer a loss. This leads to an identification of the source of business cycles with the "anarchy of capitalism", *i. e.* with the factors preventing the emergence of a cooperative behaviour.

c) After the publication of *A Contribution to the Theory of the Trade Cycle* by J. Hicks, consent grew regarding the idea that long-term growth and fluctuations could both be explained by the instability of the adjustment between output and capacity, if coupled with the idea that economic booms had to be curbed sooner or later by the ceiling of full employment. In this way, a theory of output incorporating the Keynesian principle of effective demand was reconciled with the traditional view that economic growth depends, in the long run, upon the supply of the scarce factors of production and upon their productivity. The above type of outcome can be generated by the present model only as a very particular case, following upon specific states of expectations. Thus, the rate of long-term growth g is, in general, lower than the rate of potential growth, and effective demand matters also in the long run.

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[32] A recent, if quite different, example of endogenous disequilibrium business cycles with fulfilled expectations has recently appeared in the literature. Cf. G. LAROQUE, "On the Inventory Cycle and the Instability of the Competitive Mechanism", *Econometrica*, vol. 57, n. 4, July 1989, pp. 911-935.

MATHEMATICAL APPENDIX[33]

A. *Existence of a limit cycle.*

In order to solve (15), we specify the domain of definition of the functions i( ), s( ) and f( ) and impose the standard smoothness conditions. To this purpose we introduce the open intervals E_π (a subset of R), E_δ (a subset of R_+) and E_μ (a subset of R_{++}) representing the set of admissible growth expectations, rates of depreciation, and adjustment speeds of consumer-goods output, respectively.

(16) For each $\pi \epsilon E_\pi$ ${}_\pi i(x, z)$ and ${}_\pi s(x, z)$ are defined and continuous on R^2_+ and twice continuously differentiable on R^2_{++}; they depend smoothly on π. f(z) is defined and continuous on R_+ and twice continuously differentiable on R_{++}.

Recalling that $i(x, z) = I(i(x, z) + x) + z$ and $s(x, z) = S(i(x, z) + x)$, we obtain $i_x = I_y \cdot (i_x + 1)$ and $s_x = S_y \cdot (i_x + 1)$. Thus $i_x = s_x$ at (x, z) if $I_y = S_y$ at $y = i(x, z) + x$. Assumptions (5), (6), (7) and (8) can therefore be expressed as follows:

(17) For each $\pi \epsilon E_\pi$ and each $z \epsilon R_+$ we can find a number $N > 0$ s.t. for any couple (x_1, z) for which $x_1 > N$ the following condition ${}_\pi s(x_1, z) - {}_\pi i(x_1, z) > 0$ holds true. For each $\pi \epsilon E_\pi$ and each $z \epsilon R_+$ we can find a number $x_2 > 0$ s.t. ${}_\pi s(x, z) < {}_\pi i(x, z)$ for any $0 \leq x \leq x_2$.[34]

(18) For each $\pi > \pi^*$ there exist unique real numbers y_1, y_2, with $0 < y_1 < y_2$ such that ${}_\pi i_x = {}_\pi s_x$ at (x, z) if (x, z) belongs to the union of the two separated sets $\{(x, z) \epsilon R^2_{++} \text{ s.t. } {}_\pi i(x, z) + x = y_1\}$, $\{(x, z) \epsilon R^2_{++} \text{ s.t. } {}_\pi i(x, z) + x = y_2\}$.

Assumption (18) tells us that the set of points (x, z) in R^2_{++} s.t. $i_x = s_x$, is the sum of two separated sets; each of them can be seen as the graph of a decreasing linear function, namely, $q_1(x)$ and $q_2(x)$ (see fig. 3). Linearity is shown as follows. Graph $q_j(x) = \{(x, z) \epsilon R^2_{++} \text{ s.t. } i(x, z) + x = y_j\}$ $j = 1,2$; thus graph $q_j(x)$ is a one dimensional submanifold of R_{++} and $(i_x\, dx + i_z\, dz + dx) = 0$ on graph $q_j(x)$ $j = 1,2$. Recalling that $i_z = i_x + 1$ we obtain $(dq_j(x)/dx) = -(i_x + 1)/i_z = -1$ $j = 1,2$. These graphs identify on R_{++} the three separated open regions Q_1^+, Q_2^+ and Q_3^+: $Q_1^+ = \{(x, z) \epsilon R^2_{++} \text{ s.t. } x < q_1^{-1}(z)\}$, $Q_2^+ = \{(x, z) \epsilon R^2_{++} \text{ s.t.}$

[33] The following pages draw largely upon the standard mathematical analysis of the Kaldor model (see above, n. 5); it is shown below how this analysis applies almost straightforwardly to the present model, in spite of the different treatment of autonomous expenditure. The implications of shifts in the parameter π are also considered.

[34] Cf. F. CUGNO and L. MONTRUCCHIO, *op. cit.*, p. 268.

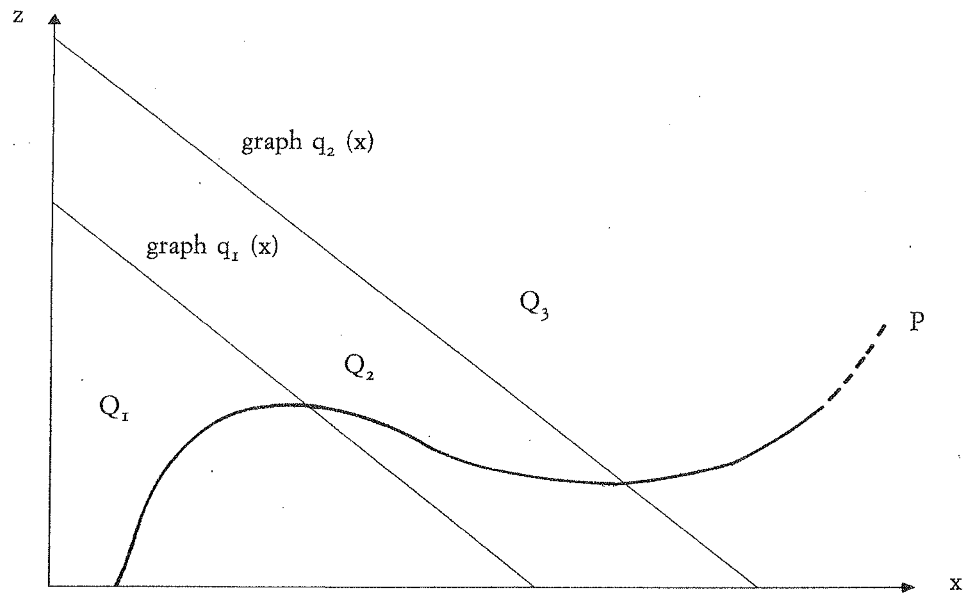

Fig. 3

$q_1^{-1}(z) < x < q_2^{-1}(z)\}$, $Q_3^+ = \{(x, z) \epsilon R^2_{++}$ s.t. $q_2^{-1}(z) < x\}$. We also define the sets Q_1, Q_2 and Q_3 by replacing R^2_+ for R^2_{++} in the definitions of Q_1^+, Q_2^+ and Q_3^+ respectively. Assumptions (7) and (8) are now expressed as follows.

(19) Let y_1 and y_2 be the output/capital ratios defined in (18) for a given $\pi > \pi^*$. There exists $z_1 > 0$ sufficiently small s.t. for each $z \epsilon [0, z_1]$ $s(x, z) > i(x, z)$ if $q_1^{-1}(z) \leq x \leq q_2^{-1}(z)$.

(20) $i(0, z) > 0$ if and only if $z > 0$.

Since $s_z = S_y \cdot i_z$, as long as the marginal propensity to save is positive and lower than one, we can impose:

(21) ${}_\pi i_z > {}_\pi s_z$ everywhere on R^2_{++}, for all $\pi \epsilon E_\pi$.

In R^2_+ we define the following sets:

$$D = \{(x, z) \epsilon R^2_+ \text{ s.t. } i(x, z) = \delta\}$$
$$B = \{(x, z) \epsilon R^2_+ \text{ s.t. } i(x, z) - \delta = f(z) + \pi\}$$
$$P = \{(x, z) \epsilon R^2_+ \text{ s.t. } i(x, z) = s(x, z)\}$$

D is the set of points in R^2_+ where the capital stock is constant, since gross investment equals depreciation. It is the graph of the strictly decreasing function $d(x)$, twice continuously differentiable on the interior of its connected domain, as a consequence of (16). Assumption (20) assures us that $d(x)$ is defined and strictly positive at $x = 0$. As shown in fig. 4, set D divides R^2_+ into two half-regions: $D_1 = \{(x, z) \in R^2_+ \text{ s.t. } z > d(x)\}$, where the capital stock increases, $D_2 = \{(x, z) \in R^2_+ \text{ s.t. } z < d(x)\}$, where the capital stock decreases over time.

B is the set of points where z is constant. It is the graph of the strictly decreasing function $b(x)$, twice continuously differentiable on the interior of its connected domain as a consequence of (16). z increases on set $B_1 = \{(x, z) \in R^2_+ \text{ s.t. } z > b(x)\}$ and decreases on set $B_2 = \{(x, z) \in R^2_+ \text{ s.t. } z < b(x)\}$. Since $f_z < 0$, as assumed in (13), it follows that $db(x)/dx > dd(x)/dx$, wherever they are both defined (see fig. 4).

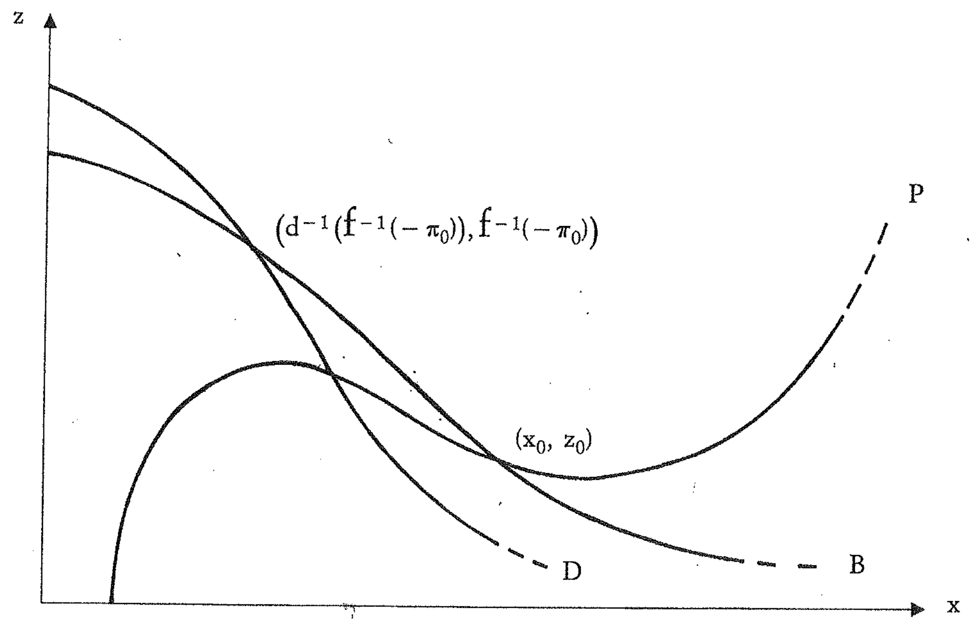

Fig. 4

With a view to making the present analysis more significant, it is useful to consider the case where the level of z sustaining a constant flow of autonomous expenditure at given π, i. e. $z = f^{-1}(-\pi)$, is lower than the level of z sustaining, at $x = 0$, a gross investment to capital ratio i( ) equal to the rate of depreciation δ; this amounts to assuming that the existence of a positive rate of long-term growth is not merely explained by conditions affecting autonomous expenditure. When the above holds true, $d(0) > b(0)$ and $\{B \cap D\}$ is the point $(x = d^{-1}(f^{-1}(-\pi)), z = f^{-1}(-\pi))$ (see fig. 4). As π keeps increasing, we eventually obtain $d(0) < b(0)$, while $\{B \cap D\}$ becomes empty.

P is the set of points in R^2_+ where C_0/K is constant; it is the graph of the function $p(x)$. Before we can say something more about set P, further restrictions must be imposed on functions i( ), s( ) and on the differential system (15).

(22) For each $\pi \epsilon E_\pi$ s.t. $\pi > \pi^*$ there exists $(x_n, z_n) \epsilon Q_2^+$ s.t. $i(x_n, z_n) = s(x_n, z_n)$.

Our treatment of autonomous expenditure (in particular, the absolute value of the first derivative $f_z(z)$ is arbitrarily large for z sufficiently close to 0) implies that the condition ruling out the possibility that an equilibrium of (15) is a saddle point does not hold on the whole state space. Still, the condition can be assumed to hold, under any admissible value of π, on a subset containing the equilibrium point.

(23) For each $\pi > \pi^*$ let $z(\pi)$ be Sup. $\{z \in R_{++}$ s.t. $s(x, z) > i(x, z)$ when $q_1^{-1}(z) \leq x \leq q_2^{-1}(z)\}$; let also $Z = \cap_{\pi > \pi^*}[0, z(\pi)]$. There exists $z_2 \epsilon Z$ s.t. $[(s_x \cdot i_z) - (i_x \cdot s_z) + f_z \cdot (i_x - s_x)] > 0$ for each $\pi \epsilon E_\pi$ and each (x, z) in the Cartesian product of the sets $\{x \epsilon R_{++}\}$, $\{z \epsilon R_{++}$ s.t. $z > z_2\}$.

Taken together, assumptions (17), (18), (19) and (22) imply that the equality $i(x, z) = s(x, z)$ holds at one point in $Q_1 \cap \{(x, z) \epsilon R^2_+$ s.t. $z = 0\}$ and at no other point (x, z) s.t. $z = 0$. This implies that $p^{-1}(0)$ is well defined and $p(x) > 0$ if $x > p^{-1}(0)$; moreover, $p^{-1}(0) > 0$ as a consequence of (17).

With the addition of assumption (23), we can rule out the possibility that an equilibrium for the differential system (15) is a saddle point. More generally, the slope of b(x) is lower than the slope of p(x) wherever they are both defined, or, in other words, set B can only intersect set P from above (see fig. 4).

This observation completes the preliminaries necessary to state the relevant properties of the function p(x) and of its graph P (see fig. 3 and 4). p(x) is defined and continuous on $\{x \geq p^{-1}(0) > 0\}$ and (twice) continuously differentiable on $\{x > p^{-1}(0)\}$. p(x) increases at x if $(x, p(x)) \epsilon Q_1^+ \cup Q_3^+$ and discreases at x if $(x, p(x)) \epsilon Q_2^+$; for any arbitrarily-large real number M there exists x_M s.t. $p(x) > M$ if $x > x_M$ (as a consequence of assumption (17)). The following proposition can now easily be proved:

(24) If assumptions (13), (14) and (16) to (23) hold true, there exists $(x_e, z_e) \epsilon R^2_{++}$ s.t. $z_e = p(x_e) = b(x_e)$. P and B have no other intersection on R^2_+.

To ensure that the equilibrium point (x_e, z_e) can be unstable for appropiate parameter values, we assume the following:

(25) Let $\pi_0 \epsilon E_\pi$ and $\delta_0 \epsilon E_\delta$; at the unique equilibrium point $(x_0, z_0) = (x_e(\pi_0, \delta_0), z_e(\pi_0, \delta_0))$ $i_x(x_0, z_0) > s_x(x_0, z_0)$.

By considering the linear part of (15) at (x_e, z_e) the following proposition is obtained:

(26) The equilibrium point (x_e, z_e) of (15) is unstable if, and only if, $\mu \cdot (i_x - s_x) x_e > (i_z - f_z) z_e$ at (x_e, z_e).

It is easy to see that the inequality in (26) is satisfied, for a sufficiently high μ, if (25) holds true. In such conditions, application of the Poincaré-Bendixon theorem yields the existence of a limit cycle in the phase space of (15).

B. *Bifurcation points*

System (15) can be shown to be structurally stable in the sense that, if the functions i( ), s( ) and f( ) meet assumptions (13), (14), (16) to (23) and (25) for parameter values π_0 and δ_0, then functions *i( ), s( ), f( )* sufficiently close to i( ), s( ), f( ) in the appropriate perturbation space, will also meet the same assumptions for parameter values π and δ sufficiently close to π_0 and δ_0 in the parameter space $E_\pi \times E_\delta$.[35]

The qualitative study of the set of parameter values meeting condition (26) gives more information on the range of validity of the cyclic dynamics just proved. For the sake of simplicity we consider a given fixed value of the parameter δ, $\delta = \delta_0$.

As a preliminary step we analyze the values of π meeting assumption (25), *i. e.* the set $G(\delta_0) \equiv \{\pi \epsilon E_\pi \text{ s.t. } (x_e(\pi, \delta_0), z_e(\pi, \delta_0)) \in Q_2{}^+\}$. Notice, in passing, that (x_e, z_e) does not depend on μ.

(27) $G(\delta_0)$ is open, bounded and connected.

Openness follows from the stability of assumption (25). Now observe that the steady state consumption ratio x_e, investment ratio $i(x_e, z_e)$ and income ratio y_e increase monotonically with π, if $i_x \geq s_x$,[36] and hence if π

[35] The main difficulty concerns the proof of stability of assumptions (23) and (25). It can be obtained by following the suggestions given by F. Cugno and L. Montrucchio, *op. cit.*, pp. 273-275.

[36] At the equilibrium point (x_e, z_e), corresponding to a point (π, μ, δ_0) in parameter space, the following conditions hold true:

(27.1) $${}_\pi i(x_e, z_e) = {}_\pi s(x_e, z_e) = f(z_e) + \pi + \delta;$$

By differentiating (27.1), we can express dx and dz as functions of $d\pi$ as follows:

$$dx = [(i_z - s_z) + s_z i_\pi - i_z s_\pi - f_z(i_\pi - s_\pi)]\, d\pi \cdot [1/(s_x i_z - i_x s_z) + f_z(i_x - s_x)]$$
$$dz = -[(i_x - s_x)(1 - i_\pi) + i_x(i_\pi - s_\pi)]\, d\pi \cdot [1/(s_x i_z - i_x s_z) + f_z(i_x - s_x)].$$

If we define the coefficients multiplying $d\pi$ in the expression for dx and dz as α and σ respectively, we can write:

(27.2) $$dx = \alpha d\pi,\ \alpha > 0; \qquad dz = \sigma d\pi,\ \sigma < 0 \text{ if } i_x \geq s_x.$$

To derive the restrictions for α and σ, recall assumption (23), and recall also that ${}_\pi s(x, z) = S({}_\pi i(x, z) + x, 1)$; since ${}_\pi i(x, z) + x = Y/K \equiv y$, we obtain:

(27.3) $$s_x = S_y(i_x + 1) \qquad s_z = S_y i_z \qquad s_\pi = S_y i_\pi.$$

$\in C(G)$.[37] Taking into account (6) and (18), we conclude that $G(\delta_0)$ is bounded and connected.

Let $S(\delta_0)$ denote the set of critical points, given δ_0, *i. e.* the points in $G(\delta_0) \times E_\mu$ such that, at the corresponding equilibrium point $(x_e(\mu, \pi, \delta_0), z_e(\mu, \pi, \delta_0))$ in phase space, one obtains:

(28) $$\mu(i_x - s_x)x_e = (i_z - f_z)z_e.$$

The set $S(\delta_0)$ can be shown to be a smooth one-dimensional manifold. Differentiating (28) and imposing convenient restrictions on the functions $S(y) \equiv S(Y/K, 1)$, $i(x, z, \pi)$ and $f(z)$,[38] we obtain the following qualitative description of $S(\delta_0)$ (see fig. 5).

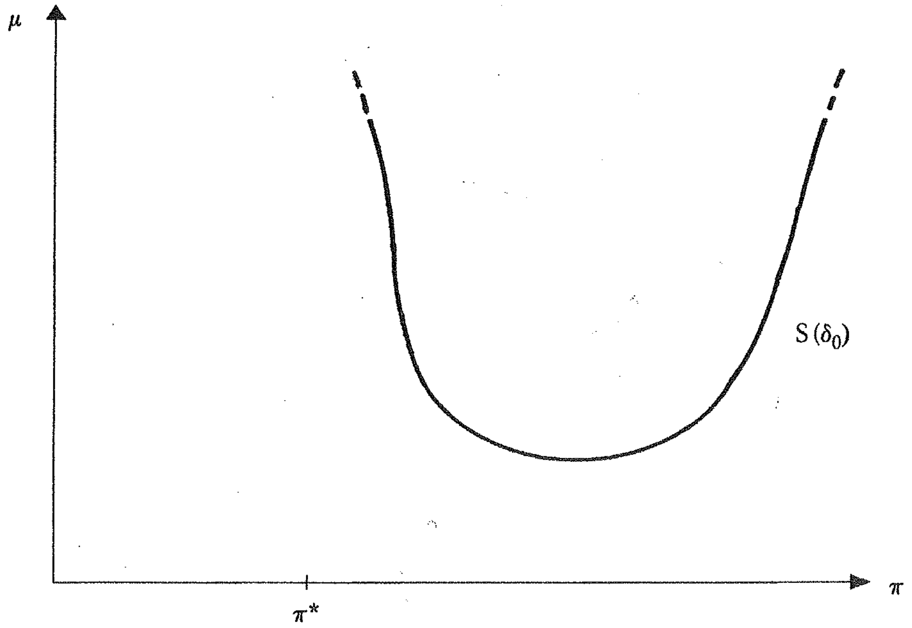

Fig. 5

The shaded region gives a qualitative description of the set $[(\pi, \mu) \in G(\delta_0) \times E\mu$ s.t. $(x_e(\pi, \mu, \delta_0), z_e(\pi, \mu, \delta_0))$ is unstable]; $S(\delta_0)$ is the set of critical points.

Substitution of the right-hand sides for the left-hand sides of (27.3) in the expressions defining α and σ proves (27.2). Taking into account (27.1) and recalling that $f_z < 0$, we conclude that ${}_\pi i(x_e, z_e)$ and y_e increase with π, if $\pi \in G$.

[37] $C(G) \equiv$ closure of G.

[38] We impose: $S_{yy} \approx 0$ (*i. e.* the savings function $S(Y, K)$ is nearly linear in Y); $f_{zz} \approx 0$ if $z > z_2$ (where z_2 is defined in (23); $\alpha > -\sigma$, which requires $i_\pi < 1/[1 + f_z(S_y^{-1} - 1)]$.