political economy *Studies in the Surplus Approach*

volume 6, numbers 1-2, 1990

Part III

Convergence to Strange Long-Period Positions

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Convergence means dynamical stability and it has usually meant asymptotic convergence to a fixed point, either monotonically or cyclically. Hence one could discuss relative fixed prices and outputs with the assumption that the economy was either there or approaching them. These fixed points obviously must shift gradually over time, as a result of technical change and demography.

Much of economic analysis has therefore been concerned with stable equilibrium to a fixed point. Our range of possible dynamics was dramatically enlarged about a century ago by Poincaré's idea of a limit cycle, but with little effect on economics. Thus Kalecki half a century ago, produced the first mathematical model (linear) of economic cycles, and was criticized by Frisch for choosing a point in his parameter space on the boundary between stability and instability, thus achieving a kind of neutral stability to explain the continued existence of cycles for a couple of centuries. Frisch maintained that the problem had to be solved by stability to a fixed point, but kept alive by exogenous shocks. This had the added advantage of explaining the evident irregularity of the cycles. Frisch like Kalecki, Hansen, Samuelson, Metzler and others, formulated a linear system, but he should have known better, since already a decade earlier, van der Pol had solved the problem by using a cubic, *i.e.* a nonlinear, relation, thus achieving an explanation of limit cycles. This formulation uses an unstable fixed point but has a nonlinearity for large movements, which bifurcates the system from local instability to global stability for large movements.

This basic discovery continued, until recently, to have no influence whatsoever on economics — with the curious exception of a brilliantly original, quite unmathematical article by the late Lord Kaldor. Then in the 1960's one E. N. Lorenz, a pupil, of Birkoff, made the dramatic discovery of 'strange attractors' or chaos (significantly with the help of computers). This constituted a remarkable addition to our ways of thinking about complicated, irregular events, like the behaviour of a bush or tree in the wind. Such a nonlinear model is stable not to a fixed point, nor to a limit cycle, but to a determinate region within which it can have an unlimited variety of irregular trajectories, monotonic and cyclical, stable and unstable. Since it is a deterministic system, which is, nonetheless, unpredictable, it seems to imply a logical and mathematical contradiction. Naturally it hass aroused much interest and analysis mathematically, but very little economically.

It must be added that there is as yet little substantial empirical economic evidence of this phenomenon in economic statistics. Nonetheless, because of the superficial resemblance of chaotic time series to economic time series, I remain convinced that nonlinear attractors are a significant addition to our understanding of economic dynamics. The problem is a difficult one, since as a result of chaotic theory, now there are two, not one, possible explanations of the highly erratic bahaviour of economic time series — exogenous shocks and strange attractors. It is difficult to determine how much of the irregularity is attributable respectively to each source, since the shocks, being exogenous, are unspecified and hence, can, so to speak, explain anything, and hence everything.

Therefore, in spite of its newness and somewhat feeble empirical support, I find this fascinating, and obviously rather basic conception immensely attractive and a potentially powerful and appropriate tool for economic analysis. Consequentially what I propose to do is to give some examples to illustrate how it could function in an extremely simplified, rather abstract economic model.

I take a simple, linear cycle model stated in ratios, hence independent of scale, which allows it to be a growth cycle. v is the ratio of employment to a constant labour force; u is the proportion of wages in net national product, which consists solely of wages and profits:

$$\dot{v} = -du + fv$$
$$\dot{u} = +hv,$$

with f representing a destabilizing element, *e.g.* the accelerator; the model then is an unstable cycle. To this is added a dynamically variable control parameter, z, in the manner of the so-called Rössler Band. The simplest economic explanation of z is that it represents the variation in net public deficits or surpluses: z negative represents a deficit with a positive effect on output and employment, and conversely for a surplus. This is because of the familiar fact that a large part of public expenditure is substantially independent of tax receipts (*e.g.*) administration, armed forces, police, and may even vary inversely, *e.g.* unemployment benefits and other public assistance). v is measured in deviations from a point, v^*, of 90%. Then $\dot{z} = b + gz(v - c)$, so that, if $c = 0.05$, when unemployment is less than 5%, public surpluses are increasing, leading to a progressive downward pressure on employment and output; with unemployment greater than 5%, the opposite upward pressure occurs.

Given an unstable cycle, the usual solution, following Poincaré and van der Pol, is to assume upper and lower nonlinearities, yielding global stability and at least one closed, stable, limit cycle. Just as Poincaré generalized the concept of possible equilibrium from a fixed point to a closed curve of motion, Lorenz succeeded in generalizing the closed curve to a bounded, closed region, in which an astonishing variety of ever-changing, aperiodic motions can occur in a seemingly erratic fashion. The upper and lower nonlinearities are replaced by a single, dynamical control parameter which provides, as necessary, either growing downward pressure or a growing upward one.

Introducing structural change in the form of a logistic growth over 50 years in innovative capacity, one gets, for a variety of initial conditions, the behaviour shown in Fig. 1. Chaotic attractors exhibit a great range of types of motion, dependent on initial conditions, so that they can be adapted to whatever is the degree of endogenous irregularity in economics. To demonstrate this, it is helpful to use the simplest, dimensionless model, thus:

$$\begin{aligned} \dot{v} &= \quad -u \quad -z, \\ \dot{u} &= \quad +v \quad +au, \\ \dot{z} &= \quad +b \quad +z(v \quad -c). \end{aligned}$$

My aim is to illustrate the gradual onset of chaotic solutions as one parameter, c, is varied (with $a = b = 0.20$). With $c = 2.2$, the result is a single limit cycle (Fig. 2); it is asymptotically stable but in an unusual way for an initial condition outside the cycle. This behaviour gradually complicates as c is increased. For $c = 3.0$, the periods are doubled (Fig. 3). To illustrate the period doubling approach to chaos, an increase of c to 3.7 doubles the doubled periods (Fig. 4). In Fig. 5 increasing c to 4.2 has further increased the number of succeeding cycles. With $c = 5.0$ (Fig. 6) there are a great many bands, each containing a subset of slightly aperiodic cycles, so that the multiplicity of bands indicates totally unperiodic behaviour, which results in bounded but highly irregular trajectories in time. Higher values of c would substantially fill the whole interior of the bounded region. Thus there is convergence to an infinity of long period positions. This stable bounded region of ever varying economic quantities seems to me to give a satisfactory conception of the relation of short-period market prices or outputs to long-period prices or outputs. Short periods breed succeeding short periods and the collection of them constitutes the long period. Thus it is a unified theory of both short and long run positions, which is endogenous and deterministic but unpredictable.

Without independent knowledge, one could not extract the model from the time series it generates, nor could one predict the future from the past. Thus though the end state seems quite bizarre, it has been produced by successive steps in varying a single parameter, which has dramatically illustrated an astonishing generalization of stable motion: from a fixed point to a fixed motion and finally to a whole bounded, attracting region of non-wandering, non-repeating trajectories. Consequently, now one has always to consider that there may be two, not just one, possible sources of erratic behaviour in economic time series.

To come closer to the reality of long period positions, the model needs to incorporate innovational investment in new capacity, producing the rising productivity. The complete, aggregative model then becomes

$$\begin{aligned}
\dot{v} &= -du - ez,\\
\dot{u} &= +hv + fu,\\
\dot{z} &= +b + gz(v - c),\\
\dot{q}/q &= (-du - ez)/(v + 0.90 + m(j + nv)(1 - sk),\\
\dot{k} &= (j + nv)k(1 - sk),
\end{aligned}$$

where k represents new, innovative capacity. Choosing plausible parameter values and appropriate initial conditions, the results are shown in Fig. 7.

The two variables, u and v, are shown as functions of time under the influence of a 50 year logistic of innovations in Fig. 8. The strongly erratic trajectories of both output and employment are clearly exhibited in both Fig. 7 and in Fig. 8.

Employment is limited by the assumption of a constant labour force, but rising productivity produces a variable growth rate of output.

These examples appear to me to exhibit the generic character of the non-repeating figures one finds in economic time series. Whilst awaiting more convincing empirical evidence of such nonlinear, endogenous irregularity, I believe that this kind of behaviour is important in economic analysis. It is, of course, essential to include also the perturbations of exogenous shocks.

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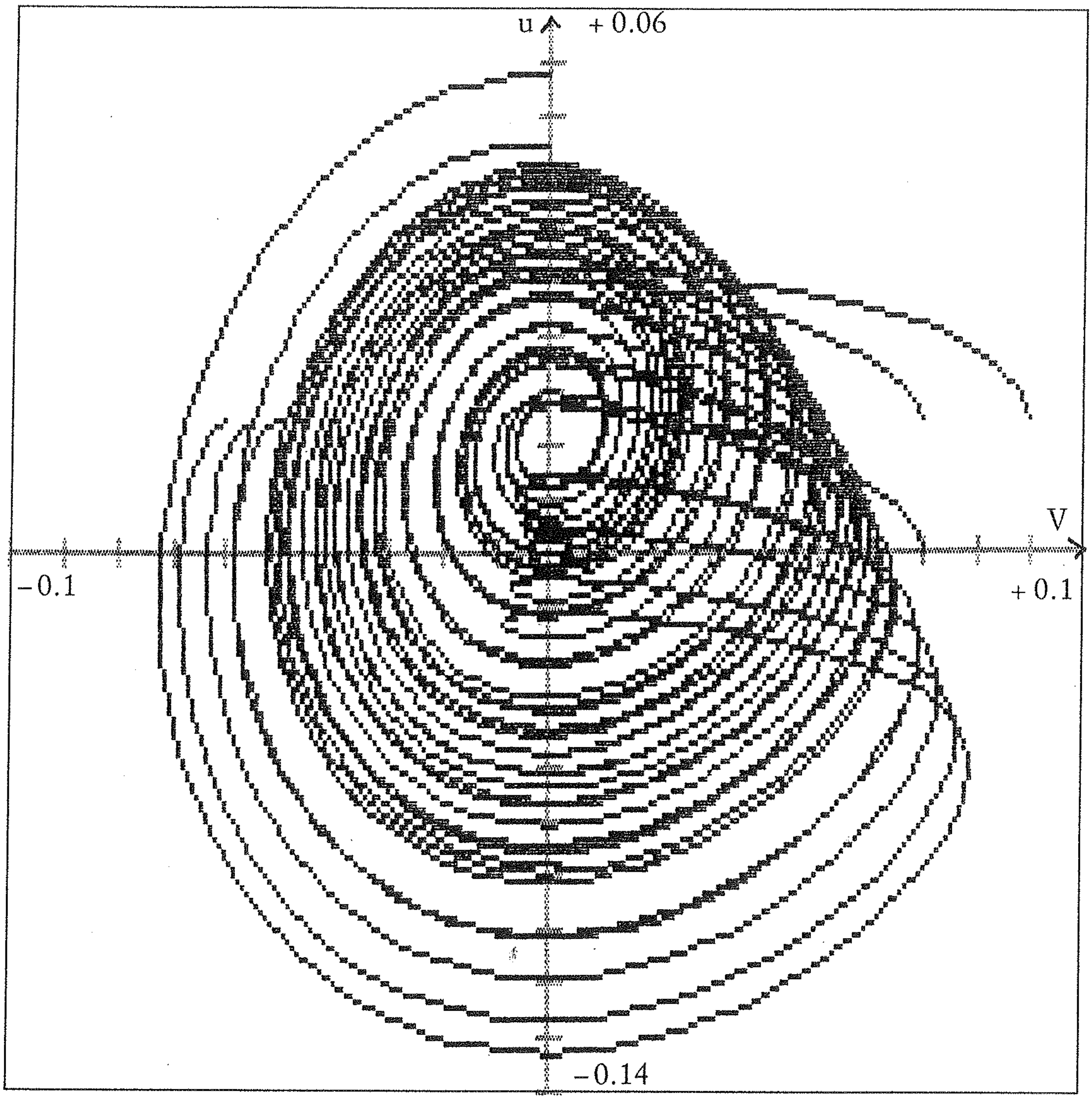


Figure 1

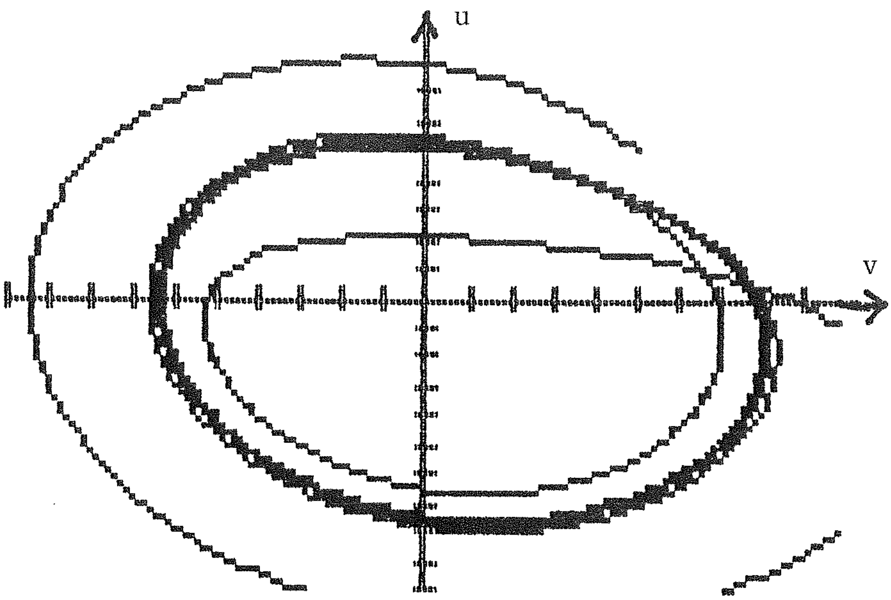


Figure 2

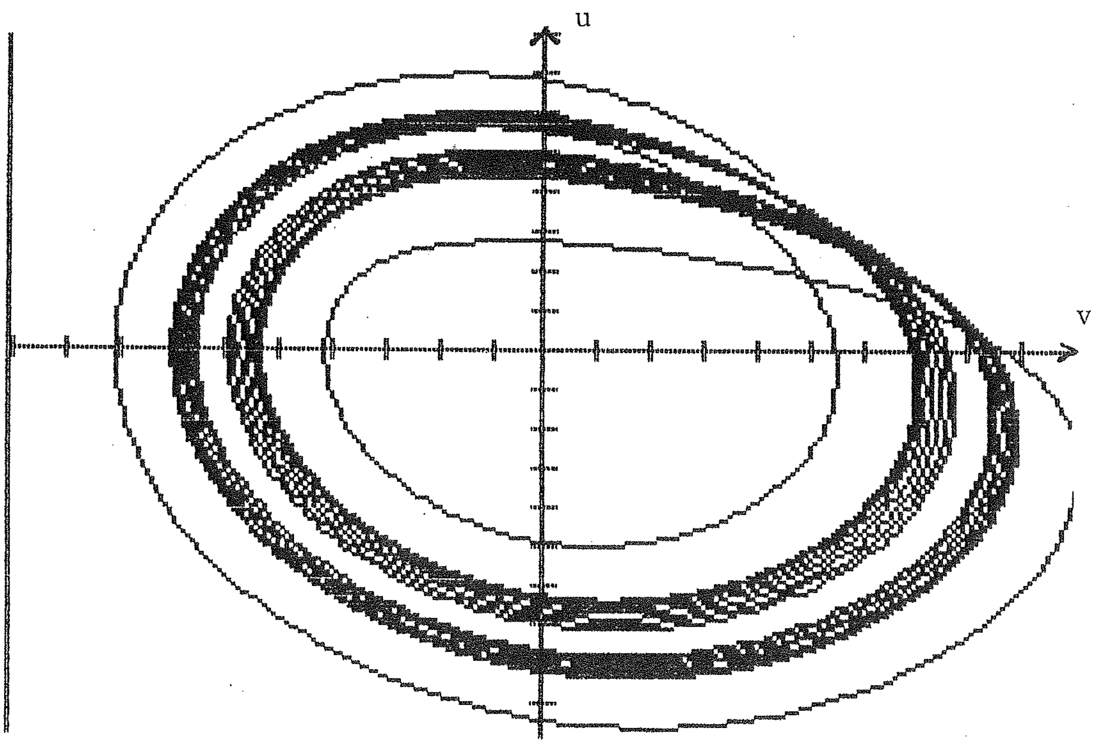
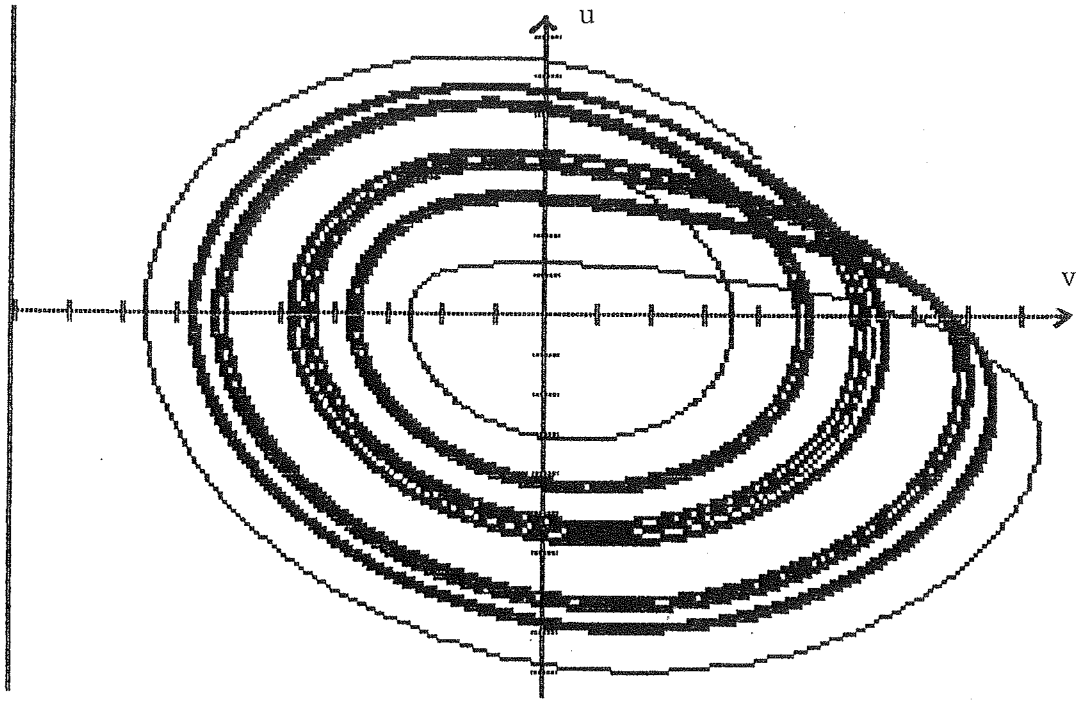


Figure 3



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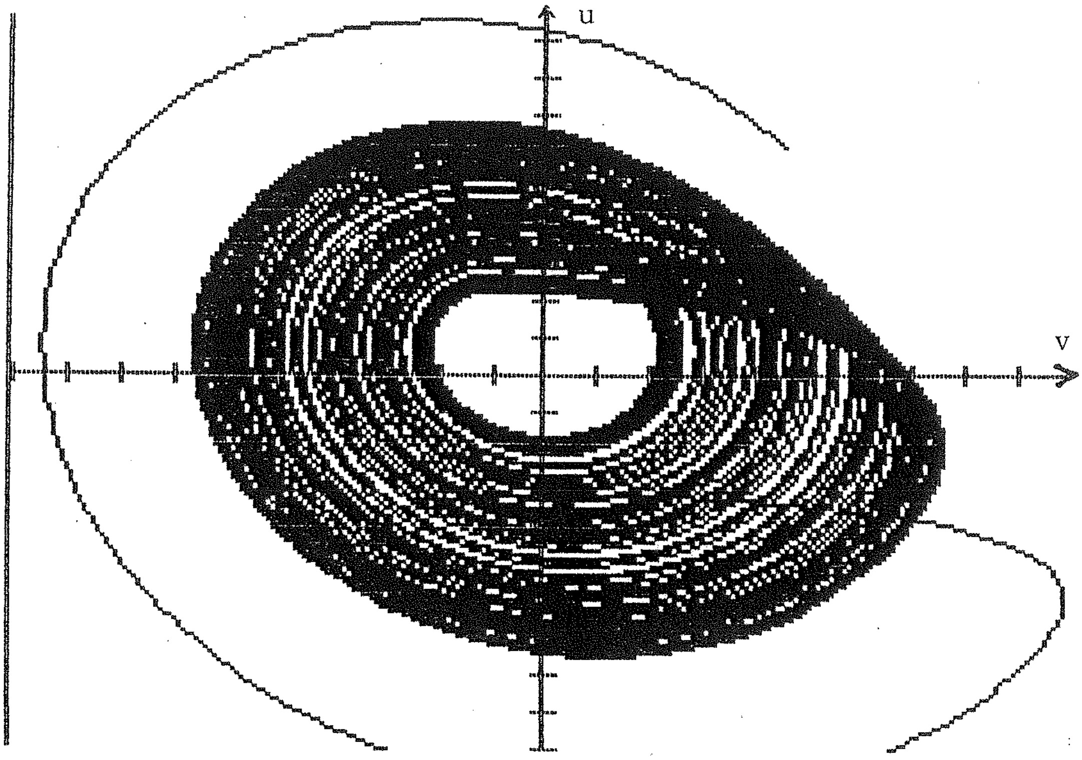

Figure 5

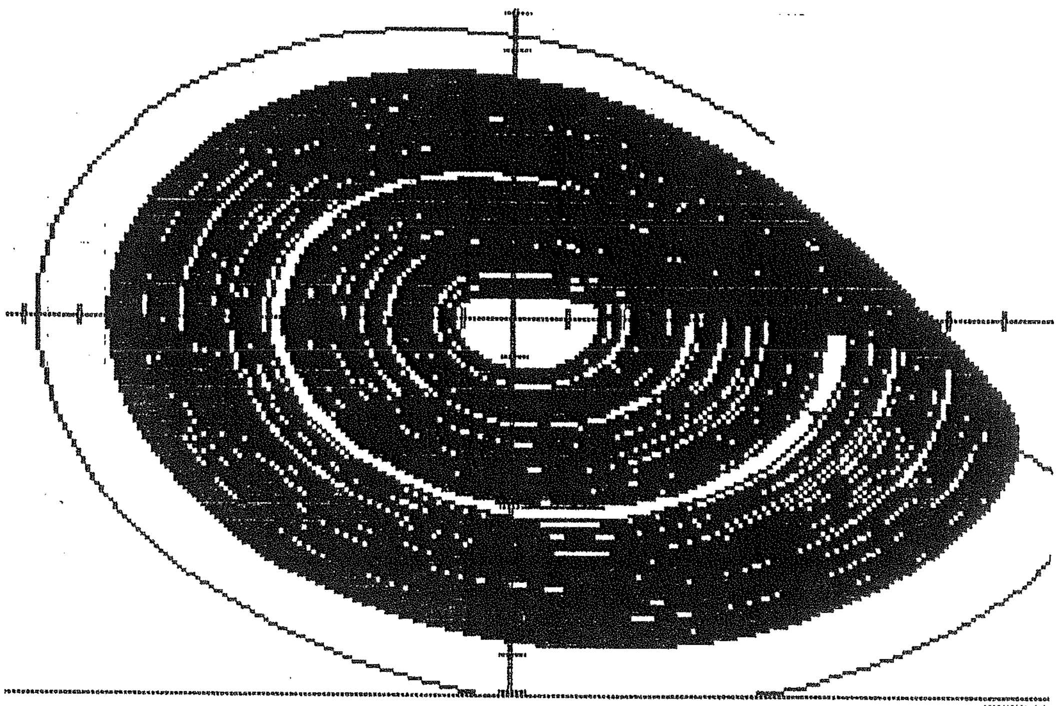
Figure 6

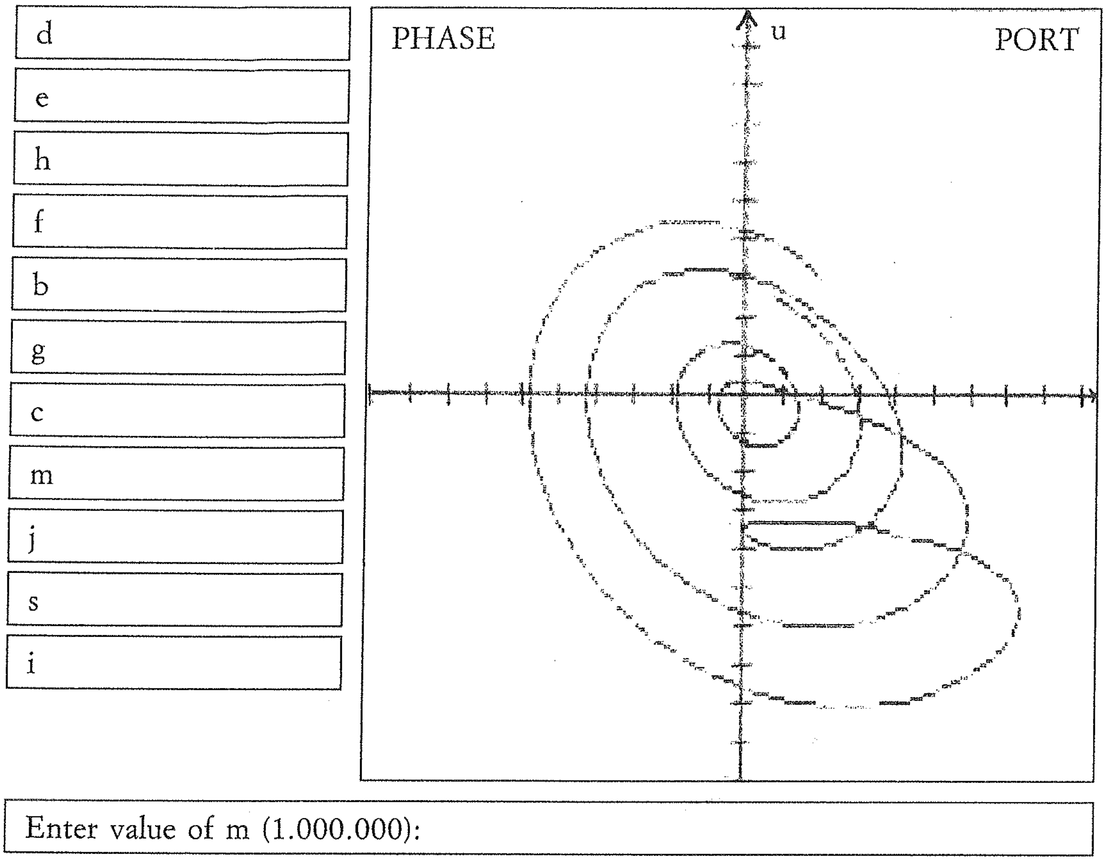

Figure 7

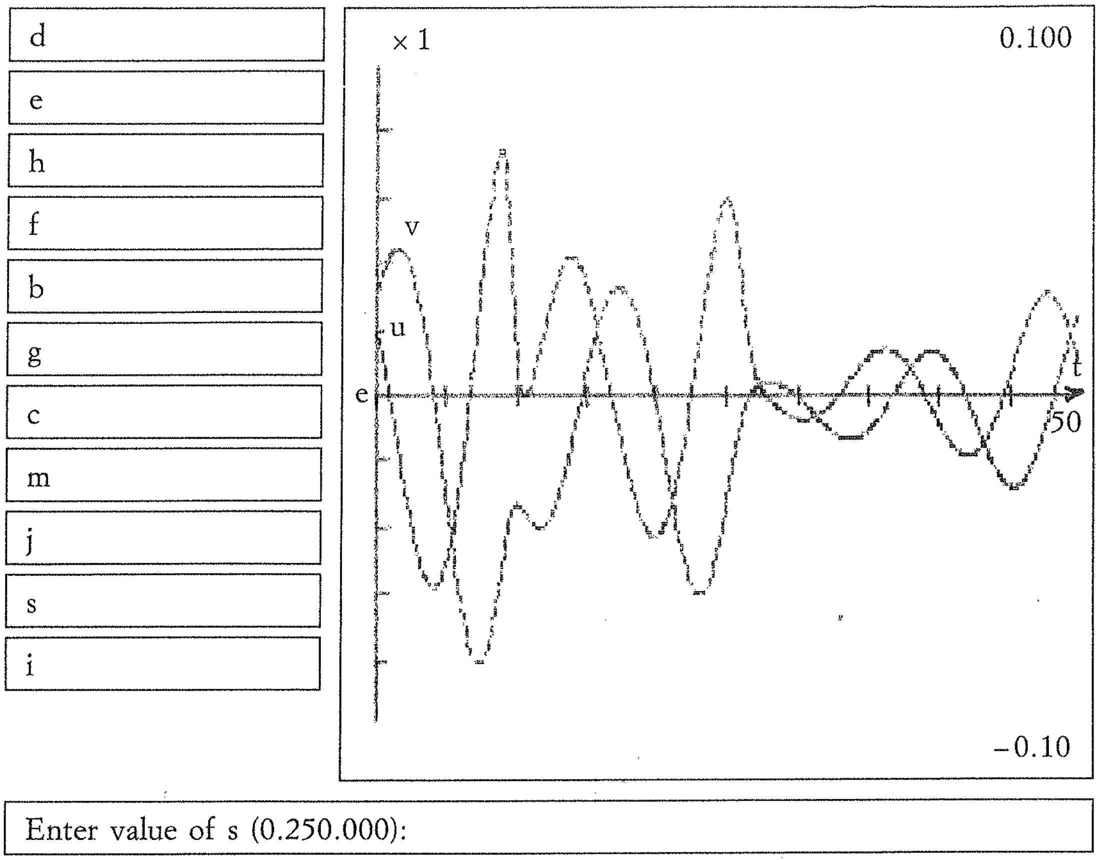

Figure 8