political economy Studies in the Surplus Approach

volume 6, numbers 1-2, 1990

On Composite Market Dynamics: Symultaneous Microeconomic Price and Quantity Adjustments

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I. INTRODUCTION[1]

In modern neoclassical theory since Hicks (1939) and Samuelson (1947), microdynamic adjustment processes have, in general, been stylized through price adjustments. The 'law of excess demand' determines the law of motion of prices. This price dynamics has been demonstrated to be asymptotically stable under the assumption of gross substitutes or the weak axiom of revealed preference.[2] Usually, the study of this dynamics is conducted by means of the celebrated price-tâtonnement. In this proposed mechanism, quantities (*i.e.* supply and demand), in contrast to prices, are assumed to adjust infinitely fast to every new price vector. In particular, there is no dynamics formulated for the adjustment of supply.

Though the above price adjustment process was originally interpreted as Walras' contribution to economic dynamics (Yaffe 1967), recently it has been recognized that Walras had also put forward a more classical oriented "disequilibrium production model" (Walker 1987), where market dynamics is based on two laws: the 'law of excess demand' and the 'law of excess returns'.[3] This two-fold adjustment process has been called "cross-dual" dynamics by Morishima (1976, 1977), or cross-field dynamics by Goodwin

[1] This paper draws on joint work with PETER FLASCHEL, whom I want to thank for helpful discussions. A related version of the paper was presented jointly with PETER FLASCHEL at an Econometrics Society session at the ASSA meeting in Atlanta, December 1989. Comments by RICHARD GOODWIN and MICHIO MORISHIMA are gratefully acknowledged. This version of the paper was prepared while I enjoyed the hospitalisty of the Economics Department at Stanford University. I want to thank DON HARRIS for stimulating discussions and for helping to arrange the visit to STANFORD. I also want to thank participants of the conference for helpful discussions and RAUL ZAMBRANO for assistance in the computer study.

[2] Cf. ARROW and HURWICZ (1958). For a summary cf. HAHN (1982). Recently, due to the work of Sonnenschein (1972) it has been recognized that the 'law of excess demand' proves to be unstable for a very general class of excess demand functions (cf. also the subsequent work by SAARI and SIMON (1978) and JORDAN (1983), which is based on those results).

[3] Extensive verbal formulations of this two-fold dynamics can be found in many classical writers in economics, for example A. SMITH (1976, ch. 7), D. RICARDO (1951, ch. 4), K. MARX (1967, ch. 10), L. WALRAS (1977, chs. 12 and 18) and A. MARSHALL (1947, chs. 3 and 5).

(1970). Accordingly, the dynamics of market systems can be formulated as follows: (i) the output of a commodity is expanded or reduced (through entry or exit of firms) whenever the excess of price over cost[4] is positive or negative ('law of excess returns'); and, (ii) the price of a commodity is raised or lowered whenever there is an excess demand or supply on the market ('law of excess demand'). Since this type of dynamic process is, by and large, built on the classical tradition in economics, we will call it classical cross-dual dynamics.

Mathematical formulations of such dynamics have been provided from the perspective of Walrasian general equilibrium theory in Beckmann and Ryder (1969) and Mas-Collel (1974, 1986), who discuss the stability of the two components of the above dynamics. Different variations of more classical versions of the above dynamics can be found, for example, in Goodwin (1953, 1970, 1988), Goodwin and Punzo (1986), Morishima (1960, 1976, 1977), Duménil and Lévy (1987a, 1987b), Franke (1987), and Flaschel and Semmler (1986, 1987).

Such formulations of short-run dynamic adjustments are considered unsatisfactory from recent macroeconomic perspectives, particularly in the Keynesian tradition. In Keynesian economics another dynamic process has been favored. This type of process has been called "dual dynamics" by Morishima (1976, 1977). The dynamic process can basically be stylized as follows: (i) quantities change due to excess demand (the output reaction of already established firms) and (ii) prices change proportional to the difference of (marked-up) costs and prices.[5] It has been maintained that these ideas can already be found in Keynes.[6]

The quantity adjustment process has become an essential element in non-Walrasian models on quantity rationing and disequilibrium analysis. The price adjustment has been elaborated by many Keynesian (or New Keynesian) theories such as the early theory of entry-deterring pricing, as well as new theories of price adjustment based on the imperfect competition/imperfect information framework. For the most part, some kind of mark-up pricing is involved to provide a justification for the above

[4] Cost can also include normal profit or the opportunity cost of capital. In some models it is interpreted as average cost (Goodwin 1953), in others as marginal cost (cf. BECKMANN and RYDER, 1969; MAS-COLLEL, 1986).

[5] Economies with such properties have been called "fixprice" economies by (Hicks (1965: 82). In the fixprice system, imbalances of supply and demand cause quantities to change and prices respond to the discrepancy between the marked-up costs and current prices (cf. HICKS, 1965:82; KALDOR, 1985; TOBIN, 1983).

[6] Though there is considerable doubt whether the above dual dynamics can be found in Keynes' 'General Theory', the 'Keynesian Revolution' is usually associated with it as already Hicks (1965:77) mentions. Leijonhuvud (1968:24) goes a step further than Hicks and suggests that the 'General Theory' represents a "systematic analysis of the behavior of a system that reacts to disturbances through 'quantity adjustments' rather than through price-level or wage-rate adjustments".

price dynamics.[7] It might be fair to state, however, that recent microeconomic theories of quantity and price adjustments do in rare cases put forward dynamic formulations of such adjustment processes.

There is though already an older Keynesian oriented tradition which has put forward dynamic versions of the aforementioned two adjustment processes. Early dynamic formalizations of Keynesian quantity adjustments, decoupled from a corresponding type of price dynamics, were given for Leontief-systems by Jorgenson (1960) and others in the form of the so-called dual instability theorem which assumes full utilization of capacity and a perfect foresight path of prices.[8]

Proper mathematical reformulations of such a dual dynamics of the Keynesian type can be found in the work of Morishima (1976, 1977), Goodwin (1970, 1988), Aoki (1977), and Fukuda (1975) (cf. also Mas-Collel, 1986, for a slightly different version). Here, for the most part, the assumptions of full utilization of capacity and perfect foresight are dropped when a stability analysis of the dynamics is provided.

Given the above two traditions in quantity and price dynamics, the classical cross-dual and the Keynesian dual fashion, a natural way to overcome deficiencies of each and to enrich our view on market adjustment processes is to integrate the two types of quantity-price adjustments into one unifying approach. Empirically it seems to be appropriate to maintain that the two types of adjustment processes are operating simultaneously, with the possibility, however, of different adjustment speeds.[9] Our paper therefore suggests a composite dynamics of price-quantity adjustments integrating the cross-dual and the dual dynamics in one dynamic system, and studies the dynamic properties of the aggregate system.

In section II a composite market dynamics will be sketched that both adds realism to our study on market dynamics and can be justified on the basis of some microeconomic considerations. Composite or aggregated microeconomic adjustment processes have originally been introduced by Flaschel and Semmler (1989a, 1989b). There, however, the stability properties of composite market dynamics were studied without obtaining general results. It was shown there that traditional methods of stability analysis can demonstrate stability for the behavior of such composite dynamic systems only under very restrictive assumption concerning the

[7] The more recent theories in particular attempt to explain the sluggishness in price adjustments compared to faster output adjustments when quantity imbalances prevail in markets. These theories are mostly based on the imperfect competition/imperfect information framework (cf. HALL, 1986; ROTEMBERG and SALONER, 1986; STIGLITZ, 1984; and GREENWALD and STIGLITZ, 1989). For a recent literature survey on wage and price rigidities cf. GORDON (1990).

[8] Starting with this early contribution, dynamic models with saddle-point instability became dominant in the perfect foresight 'rational expectations' theory.

[9] In this context one can cite he work of, for example, GORDON (1983) and TAYLOR (1980, 1986), who show that both price and quantity adjustmetns are empirically occurring simultaneously, though, as GORDON (1990) demonstrates, there are diversities across industries, time and countries.

reaction coefficients. New methods need to be explored to study the stability of composite systems. In section III, such a stability method in the tradition of Liapunov's direct approach, first explored in Flaschel and Semmler (1988), will be employed for studying the dynamics of composite systems. This method works with vector Liapunov functions and shows how conclusions may be drawn with respect to the aggregate or composite system. In section IV then, computer studies explore some further conjectures and suggest (in combination with the experience from many eigenvalue computations) that stability regions with regard to adjustment speeds may be much larger than we are able to prove analytically. On the other hand, a large class of counterexamples to stability are demonstrated to exist by employing eigenvalue studies for randomly generated matrices. Experiencing the unstable cases we then propose a sensible economic mechanism which, when introduced into the composite dynamics, will give rise to stability or bounded fluctuations. Additionally introduced stabilizing mechanisms, represented, for example, by derivative control terms as originally proposed in Flaschel and Semmler (1987), will be explored. The incorporated stabilizing terms, as will be briefly discussed, are of more general importance, for example to stabilize unstable excess demand functions.[10]

II. ON COMPOSITE MARKET DYNAMICS

The two directions in formulating economic dynamics, the classical and the Keynesian, have been introduced above. Here, briefly, before introducing the composite market dynamics, the stability results for the two separate systems will be sketched.

For the Keynesian[11] dynamics, it has been shown in Flaschel and Semmler (1988) that the equilibrium of the Keynesian (dual) dynamics is unique and, for the case $g, r < r^*$, also stable. The limit case $g = r = r^*$ can be shown to have an eigenvalue with a zero real part (of multiplicity 1 or 2). It also can be shown that the matrices involved there are stable Metzler-matrices and consequently also diagonalstable.[12] Stability of the Keynesian microeconomic adjustment processes is additionally shown by means of Liapunov functions in Appendix 1.

The cost of the Keynesian approach is, however, that certain problems are neglected the consequences of which are not thoroughly analyzed. The Keynesian dynamics neglects: (i) inventory movements, (ii) supply-

[10] For a more detailed analysis of a general type of a cross-dual dynamic system with derivative control cf. FLASCHEL and SEMMLER (1987).

[11] Without going into details, it is worth noting that the New Keynesian microtheory stylizes similar adjustment processes as was above discussed for the Keynesian version.

[12] Cf. FLASCHEL and SEMMLER (1989a, 1989b). For definitions of Metzler and diagonal stable matrices, see KEMP and KIMURA (1978:134ff).

constraints, (iii) effect of activity levels on employment, money wages and consumption, (iv) the effect of differentials in rates of return (actual rates measured against some normal rates of return), and (v) the effect of imbalances in demand and supply on prices. Output and price dynamics, in the Keynesian system, are two separate types of dynamics (as already implicit in the earlier dual instability theorem of Jorgenson, 1960). On the other hand, in the classical dynamics the properties (iv)-(v) are made essential ingredients for market adjustment processes.

For the cross-dual system the (unique) equilibrium is the same as the one for the dual-dynamics, however, the stability properties are more difficult to study. To the best of our knowledge no general results are obtained. As shown in Flaschel and Semmler (1989b), for the limit case, $g = r = r^*$, a Liapunov function can be utilized which will help to prove stability of the classical dynamics. One will, however, not obtain asymptotic stability with $g = r = r^*$. Moreover, an additional stabilizing term of a more classical nature can be proposed for classical dynamics — response of firms to the time rate of change of profit deviations from the norm — which will give rise to asymptotic stability of the cross-dual dynamics (cf. Flaschel and Semmler, 1987, 1989a).

Though in the cross-dual system price and quantity dynamics are formulated through interdependent subsystems, criticism might be raised from the aforementioned empirical perspective. There is indeed strong empirical evidence that price rigidities (due to adjustment cost or uncertainty connected to price changes) will predominantly give rise to output adjustment — rather than price adjustments — when imbalances of markets prevail and price adjustments may be dominantly occurring only through a mark-up pricing procedure.[13] Those types of market adjustments, stylized in the Keynesian dual dynamics, are missing in the classical dynamics, and modifications of it seem to be required.

Thus, a formulation of a composite price-quantity dynamics appear to be advisable that incorporates simultaneously both types of adjustment processes. The adjustments may, however, operate with different speeds.

Integrating the dual and the cross-dual adjustment processes into one composite or aggregate system gives the following more complete type of dynamics

$$\dot{x} = d_{11}C(g)\mathrm{x} - d_{12}C(r)'p' + q_1 \qquad (1)$$

$$\dot{p}' = d_{21}C(g)\mathrm{x} + d_{22}C(r)'p' + q_2 \qquad (2)$$

Here d_{11}, d_{12}, d_{21}, and d_{22} are diagonal matrices with positive diagonals, representing adjustment speeds. $C(r)' = ((1 + r)A - I)'$, $C(g) = ((1 + g)A - I)$ are nxn matrices with A the usual intermediate input matrix, r the

[13] Cf. the aforementioned literature on price adjustment, for example KALDOR (1985), HALL (1986, 1988), and GREENWALD and STIGLITZ (1989).

rate of return on capital, g the rate of growth, $q_1 = d_{11}c - d_{12}w$, $q_2 = d_{21}c + d_{22}w$ (d_{12}, d_{21} of the same type as d_{11}, d_{22}) c a (column) vector of final consumption goods, and w a vector of wage payments per unit of output; The vectors x, p (corresponding to c, w) as usual stand for activity levels and prices, respectively, and $\dot{x}$, $\dot{p}$ denote their time derivatives. The limit case arises when $g = r = r^*$. If we assume that $c \geq 0$, $w > 0$ than $0 < r,g < R^* - 1 = r^*$, derived from the scalar $1/R^* = \lambda_{max}(A) = \lambda_{max}(A') = 1/(1 + r^*)$ which is the maximum eigenvalue of the matrices A and A' (assumed to be indecomposable for simplicity).

A microeconomic interpretation of the above composite dynamics can be given as follows. Concerning the output reaction, formalized in (1), it is hypothesized that firms do not respond solely to imbalances of demand and supply when revising their production (and investment) decisions but that output in also scaled up (or down) according to whether the actual rates of return are above (or below) the norm or target rate r. Compared with the (Keynesian) quantity reaction to quantity imbalances the additional (classical) quantity reaction due to profitability differences may, however, be considered a slow dynamics — mainly initiated through entry and exit of firms. We might thus assume that $d_{12} < d_{11}$.

On the other hand, concerning price dynamics, in our view it also adds realism to a model of market dynamics if one assumes a two-fold process. Accordingly, in our dynamics (2) we posit that firms when they set prices follow two decision-making criteria: first, prices are provisinally set on the basis of a mark-up (or target rate of return) calculation and secondly, they are further revised, through an error correcting process, in proportion to the imbalance of demand and supply in the various markets. Price studies for large firms appear to support such a price dynamics (cf. Semmler, 1984, ch. 3).

The existence and uniqueness of the equilibrium of (1), (2) is proved in Flaschel and Semmler (1989b). For convenience, in the subsequent sections, we want to write (1), (2) in compact notation as

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} d_{11}c\,(g) & -\,d_{12}c\,(r)' \\ d_{21}c\,(g) & d_{22}c\,(r)' \end{bmatrix} \begin{bmatrix} x \\ p' \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3}$$

The dynamic properties of (3), where the different subsystems now interact, are to be studied. Note that (for $g,r < r^*$) the Keynesian dual dynamics, sketched in the diagonal of the above matrix, is asymptotically stable and the classical cross-dual dynamics, portrayed in the off-diagonal terms of the above matrix,[14] is marginally stable. One might therefore conjecture that the stability of system (3) above is easy to demonstrate. Yet stability of the simple case, with a constant coefficients' (square) matrix

[14] For a more general model of classical type with explicit stability analysis, cf. FLASCHEL and SEMMLER (1987).

as referred to above and constants g and r, is not obvious. In addition, a modified case will arise when, as will be shown subsequently, the rate of return r and the growth rate g is endogenized by using their average rates in each period in the dynamics (3).

III. STABILITY ANALYSIS OF THE COMPOSITE MARKET DYNAMICS

Several methods have been explored in Flaschel and Semmler (1989a, 1989b) to study the dynamics of system (3). All methods (diagonal dominance, quasi negative definiteness and Liapunov functions) revealed that the stability of the composite system is crucially dependent on the constellations of the reaction coefficients in the dual and cross-dual dynamics. With the method of diagonal dominance and negative definiteness of matricess only simple cases of stability could be demonstrated. The remainder of the paper, by employing vector differential inequalities and vector Liapunov functions as put forward in Siljak (1978)[15] and as already discussed in Flaschel and Semmler (1988, 1989b), will show that a composite system such as (3) will be stable if the subsystem represented by the cross-dual dynamics interacts only weakly with the subsystem portraying the dual dynamics. Subsequently, some extensions wil be explored.

The type of stability analysis which is involved here is termed connective stability. Interesting methods for studying connective stability for composed systems by a decomposition-aggregation procedure are provided by the concepts of vector differential inequalities and vector Liapunov functions as elaborated in Siljak (1978, ch.2) and summarized in Appendix I.

III.1. *Stability with Weakly Connected Subsystems*

From our above stability discussion of the separate subsystems it is clear that the problem we are facing concerning the stability of system (3) is less severe than, for example, that appearing in other cases (cf. remark 1 and 2 in Appendix I). Our composite system (3) is of the form (A1) of Appendix I with two stable decoupled subsystems $\bar{A}_{11} = d_{11}C(g)$, $\bar{A}_{22} = d_{22}C(r)'$ for which the two Liapunov functions as needed for the proof (cf. Appendix I) exist.

In order to investigate the asymptotic stability of the totally interconnected system and the interconnections allowed here we need to consider, according to the theorem in Appendix I, the following matrix:

[15] Further references on this topic are BERUSSOU and TITLI (1982), MEDIO (1987), MICHEL and MILLER (1977), and SINGH and TITLI (1979).

$$(\bar{w}_{i,j}) = \begin{bmatrix} -(\Omega_M^{1/2}(H_1))^{-1}\, 1/2\, \dfrac{1}{\Omega_M^{1/2}(H_1)} & \epsilon_{12}\,(\Omega_m^{1/2}(H_2))^{-1}\, \dfrac{\Omega_M(H_1)}{\Omega_m^{1/2}(H_1)} \\ \epsilon_{21}\,(\Omega_m^{1/2}(H_1))^{-1}\, \dfrac{\Omega_M(H_2)}{\Omega_m^{1/2}(H_2)} & -\,\Omega_M^{1/2}(H_2)^{-1}\, 1/2\, \dfrac{1}{\Omega_M^{1/2}(H_2)} \end{bmatrix}$$

$$= \begin{bmatrix} -1/2\,(\Omega_M(H_1))^{-1} & \epsilon_{12}\, \dfrac{\Omega_M(H_1)}{\Omega_m^{1/2}(H_1)\,\Omega_m^{1/2}(H_2)} \\ \epsilon_{21}\, \dfrac{\Omega_M(H_2)}{\Omega_m^{1/2}(H_1)\,\Omega_m^{1/2}(H_2)} & -1/2\,(\Omega_M(H_2))^{-1} \end{bmatrix} \qquad (4)$$

with $\epsilon_{12} = \; = \Omega_M^{1/2}(C(r)\,d_{12}{}^2C'(r))$ and $\epsilon_{21} = \Omega_M^{1/2}(C(g)'\,d_{21}{}^2C(g)) > 0$.

Sufficient for the stability of our composite system is that the Metzlerian matrix (4) is Hicksian, cf. Kemp and Kimura (1978:141ff.). Since $\bar{w}_{11}$, $\bar{w}_{22}$ are negative, we therefore have to explore only whether $\mathrm{Det}(\bar{W})$ is positive. We have

$$Det(\bar{W}) = 1/4\, \frac{1}{\Omega_M(H_1)\,\Omega_M(H_2)} - \epsilon_{12}\,\epsilon_{21}\, \frac{\Omega_M(H_1)\,\Omega_M(H_2)}{\Omega_m(H_1)\,\Omega_m(H_2)}$$

$$= 1/4\, \frac{1}{\Omega_M(H_1)\,\Omega_M(H_2)} \left[1 - \frac{(2\,\Omega_M(H_1)\,\Omega_M(H_2))^2}{\Omega_m(H_1)\,\Omega_m(H_2)}\,\epsilon_{12}\,\epsilon_{21} \right] \qquad (5)$$

In order to explore situations where $\mathrm{Det}(\bar{W}) > 0$ holds true we consider the following scalars in front of the reaction coefficients d_{11}, d_{22}, d_{12}, d_{21}:

$$\alpha_{11} d_{11},\ \alpha_{22} d_{22},\ \alpha_{12} d_{12},\ \alpha_{21} d_{21} \quad \text{with } \alpha_{ij} > 0.$$

In this case H_1, H_2 are to be substituted by H_1/α_{11}, H_2/α_{22} as can be seen immediately from the following explication (cf. Appendix I)

$$C(g)'\,d_{11}'\,H_1 + \mathrm{H}_1 d_{11} C(g) = -I,\ C(r)\,d_{22} H_2 + H_2 d_{22} C(r)' = -I.$$

The elements of matrix W are thus determined in nearly the same way as before — with the provision that scalars α_{ij} appear as multipliers at the appropriate places. It remains to be shown, therefore, that the expression in square brackets [·] in the determinant (5) becomes positive for appropriate variations of adjustment speeds. We can have

$$1 - \frac{\alpha_{11}^{-2}\,\alpha_{22}^{-2}}{\alpha_{11}^{-1}\,\alpha_{22}^{-1}}\, \frac{(2\,\Omega_M(H_1)\,\Omega_M(H_2))^2}{\Omega_m(H_1)\,\Omega_m(H_2)}\,\alpha_{12}\,\alpha_{21}\,(\epsilon_{12}\,\epsilon_{21}) > 0 \qquad (6)$$

for a suitably chosen range of α_{ij}.

Now it is obvious from (6) that for given α_{12}, α_{21}, for example, choosing sufficiently large scalars α_{11}, α_{22} will render the system asymptotically stable (or for given α_{11}, α_{22} there exist always sufficiently small α_{12}, α_{21} which will generate connective stability, as defined in Appendix I, for system (3)). This shows that the composite system will be asymptotically stable if the classical dynamics is made sufficiently weak or long-run in nature.[16] Furthermore, the expression (6) can be equivalently rewritten as follows

$$(\Omega_m(H_1)\,\Omega_m(H_2))/(2\Omega_M(H_1)\,\Omega_M(H_2))^2 > (\epsilon_{12}\alpha_{12}\epsilon_{21}\alpha_{21})/(\alpha_{11}\alpha_{22}) \qquad (7)$$

In this form it shows how the stability characteristics of the Keynesian subsystem (as they are expressed by the two Liapunov matrices H_1, H_2) must dominate the off-diagonal interaction coefficients to obtain overall stability: The larger the interaction parameters ϵ_{ij} are, the smaller the α_{ij} we have to choose to make the approach of Appendix I applicable (the α_{ij} can be set equal to one without loss of generality).

However, the above calculations also show that estimates for $\Omega_m(H_i)$, $\Omega_M(H_i, \epsilon_{ij}$ have to be developed first to obtain more than the above very general statements.[17] A problem with this approach is, however, that it is insensitive to the typical sign structure of our cross-dual interconnection. The advantage of this method over one-shot approaches is noticeable but, in the present case, still somewhat limited. It may be advisable in subsequent research to use more specific features in the interconnective (or classical) part of the composite dynamics so that its influence on the stability of the system becomes more apparent.

III.2. *Additional Stability with Derivative Control*

The remarks 1 and 2 in Appendix I indicate that one should make more use of extensions of our composite dynamics to obtain overall stability. In this regard derivative control may be a helpful device. We will indicate here briefly how these extensions can be pursued. From remark 2 in Appendix I one can derive, for our composite dynamics (3), a refined adjustment such as

$$\dot{x} = -d_{12}[C(r)'p' + \gamma C(r)'\dot{p}']$$

instead of only $\dot{x} = -d_{12}C(r)'p'$. The new term $-\gamma d_{12}C(r)'\dot{p}$ expresses the fact that the time rate of change of profit-rate differentials is also considered by firms when scaling up (or down) production as formulated in (2). Extending the dynamics (2), for example for $g = r = r^*$, by including the above term gives

$$\dot{x} = d_{11}C(g)x - d_{12}C(r)'p' - \gamma d_{12}C(r)'\,[d_{21}C(g)x + d_{22}C(r)'p']$$
$$\dot{p} = d_{21}C(g)x + d_{22}C(r)'p'$$

[16] The slow adjustment speed for the cross-dual dynamics seem to be confirmed by the large body of empirical literature on price rigidities and quantity adjustments as cited above.

[17] Cf. SILJAK (1978:110-111) for a simple numerical example in this regard. For solving the above problem Kronecker products and their application to matrix equations of type (A2) in Appendix I can be used, cf. LANCASTER and TISMENETSKY (1985, ch. 12) for details. Yet, an investigation of the numerics of the above approach is beyond the scope of the present paper.

With regard to such an extension we get for the composite system in terms of equation (A1):

$$\dot{z}_1 = A_{11} z_1 + e'_{12} e_{21} \gamma A_{12} A_{21} z_1 + e_{12} A_{12} z_2 + e'_{12} \gamma A_{12} A_{22} z_2$$
$$\dot{z}_2 = A_{22} z_2 + e_{21} A_{21} z_1. \tag{8}$$

For $g = r^*$ we have for $A_{12}A_{21}$ the expression $-d_{12}C' d_{21}C$ with this matrix being symmetric and negative definite. With $e'_{12} = e_{21} = 1$ we therefore get with $e'_{12}e_{21}\gamma A_{12}A_{21}$ an extra stabilizing term in the diagonal of (8). This new stabilizing term has to be contrasted, in its influence on the composite dynamics, with the new off-diagonal term $A_{12}A_{22} = -d_{12} C' d_{22} C'$ and its dynamic effect on the cross-dual interaction (c.f. sect. IV.2. below).

Note that the excess demand functions (as they appear since Hicks in neoclassical tâtonnement analysis) are here represented through the terms A_{11}, A_{21}. More general excess demand functions which are known to create problems for the one-sided conventional tâtonnement process are thereby made less dominant in (8) than in (3). Their influence on price dynamics (given by $e_{21}A_{21}z_1$) is now off-diagonal. It is, according to our considerations in (6) and (7), only to be properly limited (but it no longer needs to be stable itself). Their possibly destabilizing influence on the quantity dynamics is now partly compensated for through the new stabilizing term in the first part of (8).

In sum, it has been shown that by employing vector Liapunov functions sufficient restrictions on the adjustment speed of the cross-dual dynamics will always render the composite system stable. This method, as those discussed in Flaschel and Semmler (1989a, 1989b), cannot demonstrate stability independently of the adjustment coefficients d_{ij}. On the other hand, our composite system (3) has a more specific structure than assumed in the proof of stability in sect. III.1. Stability of the composite system, therefore, may prevail even with stronger reaction coefficients in the cross-dual part. We also might want to explore the stability properties for basically unstable systems when additional terms, representing derivative control, are operative. These problems cannot be discussed analytically in the context of Siljak's approach. Further computer studies are needed.

IV. COMPOSITE DYNAMICS AND DERIVATIVE CONTROL: COMPUTER STUDIES

In what follows we discuss theree types of extensions. First, simulaton studies will be discussed that show that stability of our composite dynamics prevails even if the off-diagonal reaction coefficients are increased and even if nonlinearities are introduced. The nonlinearities are represented by endogenized profit and growth rates. Second, we display simulations for a case when there are strong off-diagonal reaction coefficients in system

(3) rendering the system unstable. A larger class of unstable matrices will then be presented which were detected by employing eigenvalue studies for matrices generated by a random number generator. Third, we want to demonstrate how trajectories of the dynamics of even unstable systems (either stemming from the unstable matrices or from the off-diagonal reaction coefficients) can be stabilized by a sensible economic mechanism which gives rise to stable or bounded fluctuations.

IV.1. *Composite Dynamics: Stability Regions*

In the first type of extension we are considering a basically stable system, which remains stable for a large range of reaction coefficients, and we drop the assumption on constant growth and profit rates $g \leq r \leq r^*$. We will, however, restrict our attention to the limit case where $g_t = r_t$ fluctuates around the rate r^*.

For simulations we used system (1), (2) in time discrete form as basic model.

$$x_{t+h} = x_t + h(d_{11}C(g)x_t - d_{12}C(r)'p'_t + q^1) \qquad (1')$$

$$p_{t+h} = p'_t + h(d_{21}C(g)x_t + d_{22}C(r)'p'_t + q^2) \qquad (2')$$

with $q^1 = d_{11}c - d_{12}w$, $q^2 = d_{21}c + d_{22}w$ and where r and g constants or endogenized.[18]

To illustrate stability of a simple case, we choose a 2 dimensional system with the following A matrix and matrices of reaction coefficients.[19]

$$A = \begin{bmatrix} .35 & .55 \\ .25 & .45 \end{bmatrix}, \; d_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; d_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$d_{11} = \begin{bmatrix} .2 & 0 \\ 0 & .2 \end{bmatrix}, \; d_{22} = \begin{bmatrix} .5 & 0 \\ 0 & .4 \end{bmatrix}.$$

Whereas figure 1 represents the graphs for the time path of relative prices and relative outputs of the composite system for $1 + r = 1 + g = R^* = 1.29 = 1/\lambda_{max}(A)$, figure 2 depicts the relative price and output dynamics for the average and endogenously determined growth rate and profit rate $R_t = p_t x_t / p_t A x_t$.

Both relative prices and relative outputs exhibit asymptotic stability for the cases r^* as well as $r_t = g_t$. In the latter case the differential equation system (1), (2), or their time discrete counterparts (1′) and (2′), are nonlinear. In the simulations for $r^* = g^* =$ constant, depicted in figure 1, the reaction coefficients of the off-diagonal terms could be further increased without

[18] There exist, of course, better numerical procedures for integrating the differential equation system (1), (2) or their time discrete counter parts (1′), (2′). Our above simple procedure with step size h (and h = .2) is chosen because it might reflect better actual economic behavior of firms.

[19] Higher dimensional examples of our composite dynamics with r and g endogenized can be found in FLASCHEL and SEMMLER (1989b).

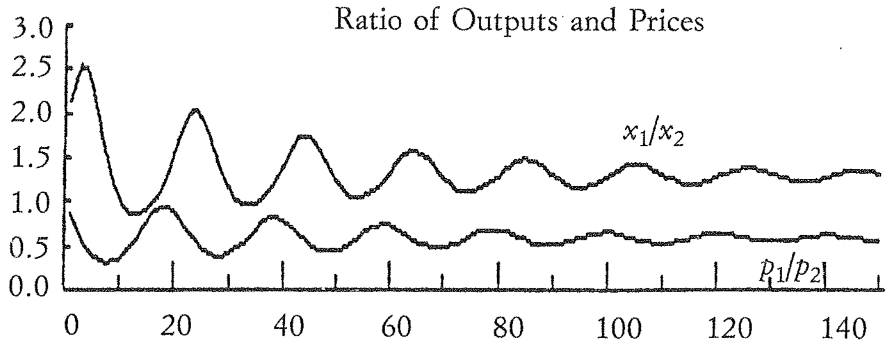

Figure 1

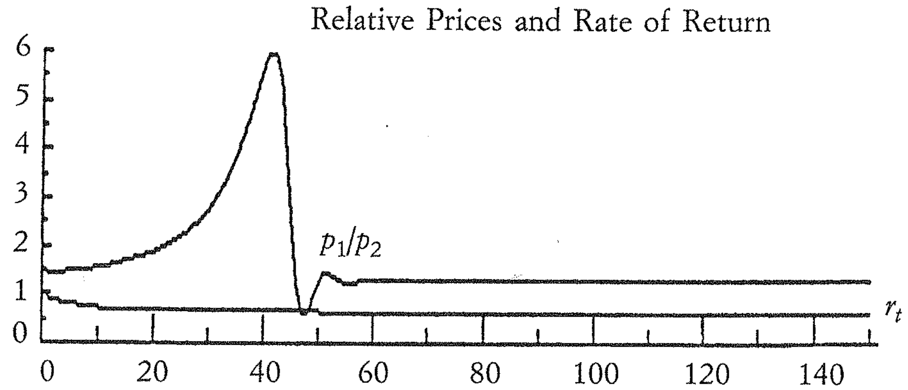

Figure 2

endangering the stability properties of the system (1′) and (2′). For $r_t = g_t$ however, the reaction coefficients for the off-diagonal terms had to be made 10 times smaller than the coefficients for the diagonal matrices in order to obtain stability. The nonlinearity introduced in (1′), (2′) easily leads to instability of the composite system.

In sum, the above (and further simulation and eigenvalue studies) support the conjecture that for proper matrices the region of stability is, at least in the case where $r = g =$ constant, much larger than indicated in the proofs of section III. The dynamics can also be of a more general type, including nonlinearities in the differential equations (1) and (2), and still generate stability. Similar results were obtained for composite systems of higher dimensions than discussed here (cf. Flaschel and Semmler, 1989b).

Next, we want to consider a larger class of systems that give rise to dynamic instability. Instability, as will be seen, occurs with or without the help of reaction coefficients.[20] We again refer to a time discrete version such as (1′), (2′), however, with constants $g \leq r < r^*$.

First we want to present a simulation result. From the large set of unstable matrices, generated by a random number generator as described below, we pick a 6*6 randomly generated matrix (cf. matrix A in Appendix II.1). The Q matrix based on A is unstable due to strong reaction coefficients for the off-diagonal terms. We find that the system has a pair of real parts of eigenvalues $\lambda_{max}(A) = 5.45$ (cf. Appendix II.1). We expect unstable trajectories.

With arbitrary $c = .03\ .03\ .03\ .03\ .03\ .03$ and $w = .01\ .01\ .01\ .01\ .01\ .01$ the system has an equilibrium at $p^* = .18\ .17\ .17\ .17\ .17\ .15$ and $x^* = .18\ .19\ .15\ .11\ .15\ .18$.

For the equilibrium vectors x^*, p^* as well as for the actual p and x we use a normalization such that $\Sigma p_i = 1$ and $\Sigma x_i = 1$, and introduce a Euclidian norm to measure the distance from the actual to the equilibrium vectors. This is undertaken in order to allow for a selective derivative control in case instability arises.

Figure 3 depicts the unstable trajectories where the off-diagonal reaction parameters are 1000 times stronger than those of the diagonal terms.[21]

[20] The question then arises how the instability can be bounded by a reasonable economic adjustment process. This problem will be pursued further below.

[21] Other unstable matrices with less extreme reaction patterns are reported below.

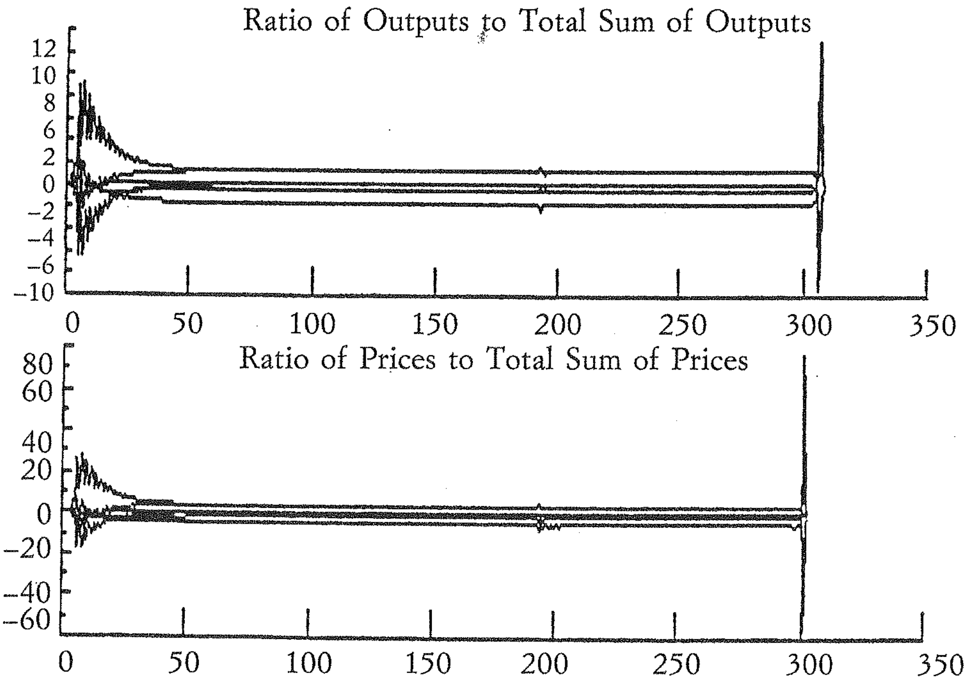

Figure 3

In order to detect a larger class of economically realistic unstable matrices, a computer program was written that test randomly generated matrices for instability. We created 12 and 6 dimensional matrices by a random number generator. All the matrices had the required properties of non-negativity and $1 < G \le\ < R \le R^*$ so that the matrices were productive with positive growth and profit rates. Matrices of the following type were tested; for simplicity it was assumed d_{ij} equal to identity matrices and γ_1, γ_2 the scalars for the reaction coefficients.

$$Q = \begin{bmatrix} \gamma_1 d_{11} C(g) & -\gamma_2 d_{12} C(r)' \\ \gamma_2 d_{21} C(g) & \gamma_1 d_{22} C(r)' \end{bmatrix} \qquad (9)$$

$G < R = R^*$

dim	γ_1	γ_2	R^*_m	R	G	NMT	NMR^*_m	real part	ratio
12	1	1000	1.5	1.5	$.68R^*_m$	500	260	170	170/260
12	1	100	1.5	1.5	$.68R^*_m$	500	268	52	52/268
12	1	50	1.5	1.5	$.68R^*_m$	500	262	1	1/262
12	1	1	1.5	1.5	$.68R^*_m$	1500	790	0	0/790

Table 1

$G = R < R^*$

dim	γ_1	γ_2	R^*_m	R	G	NMT	NMR^*_m	real part	ratio
12	1	1000	1.55	$.7R^*$	$.7R^*$	500	250	0	0/250
6	1	1000	1.25	$.85R^*_m$	$.85R^*_m$	500	390	0	0/390

Table 2

$G < R < R^*$

dim	γ_1	γ_2	R^*_m	R	G	NMT	NMR^*_m	real part	ratio
12	1	1000	1.5	$.9R^*_m$	$.7R^*_m$	500	365	87	87/375
6	1	1000	1.35	$.9R^*_m$	$.75R^*_m$	500	340	13	13/340

Table 3

In the above tables dim, R^*_m, NMT, NMR^*_m real part, ratio denote the dimension of the matrices, the minimum R^* allowed for in the random matrices, the number of matrices being tested for their set of eigenvalues, the number of matrices found with $1/\lambda_{max}$ greater than R^*_m, the number of eigenvalues with real parts greater than zero found among the matrices tested (possibly more than one pair for one matrices), and the ratio of real parts of eigenvalues greater than zero to the number of significant matrices. Moreover, as before we have $R = 1 + r$ and $G = 1 + g$.

As can be observed in table 1, with γ_2 decreasing the number of unstable matrices declines sharply, and if γ_2 becomes equal to γ_1, there is no unstable matrix any more, even if the number of matrices tested has increased to 1500. Also, for the case $G = R < R^*$, table 2, unstable matrices are not easily found. There are, however, unstable matrices for $G < R < R^*$

(cf. table 3). For the case $G = R < R^*$ there exist unstable matrices, but of a different type. Unstable matrices of that type (which are matrices with hypercycles) are pesented in Flaschel and Semmler (1989b).[22]

IV.2. *The Composite Dynamics with Derivative Control*

The above unstable cases motivate further explanation of stabilizing mechanisms. We will introduce in our dynamics (3) a derivative control as proposed in Kose (1956), Flaschel and Semmler (1987), and as briefly discussed above, and study the resulting stability properties therefrom.[23] The derivative control can operate partially or totally. Moreover, it can be employed everywhere (globally) or in certain regions of the vector field only (selectively). We explore all variants.

Case 1: Derivative Control for the Rate of Return

Introducing a partial derivative control for the differentials in rates of return as suggested in Flaschel and Semmler (1987), we can write subsystem (2′) as[24]

$$x_{t+h} = x_t + h(d_{11}C(g)x_t - d_{12}C(r)'p'_t + q_1 - \gamma d_{12}C(r)'\dot{p}'_t) \qquad (1'')$$

where the additional term is approximated by a time discrete version[25]

$$\gamma d_{12}C(r)'(p_{t+h} - p_t)'/h = \gamma d_{12}C(r)'(d_{21}C(g)x_t + d_{22}\hat{C}(r)'p'_t + q_2) \quad (10)$$

The additional term (10), as already discussed in sect. IV.2, represents firms' reaction to the rate of change of profit flows so that firms, in their output decisions, take into account not only the imbalances of supply and demand and differential rates of return but also the time derivative of the differentials of rates of return. In many circumstances such a derivative control term (cf. Flaschel, 1989) acts as a stabilizing term. Unfortunately, a proof of stability, provided in Flaschel (1989) for simpler cases, does not carry over to our system (1″), (2′), so that again we have to rely on eigenvalue computations and simulations.[26]

[22] Since they are, however, economically not very realistic, we here leave them aside.

[23] Richard Goodwin, in a comment on an earlier version of our paper, has proposed that we may allow for instability in our composite system, but we might want to be concerned with how the trajectories are globally bounded. The subsequent section will explore this suggestion.

[24] Note that, for reasons of simplicity, in the subsequent system the diagonal matrices of the reaction coefficients also include the scalars as referred to in Appendix II.

[25] For a formulation of a time continuous dynamics with derivative control term in time continuous form, cf. Flaschel and Semmler (1987).

[26] With the matrix Q as appearing in (9) we can write system (1), (2) (by dropping the constant term since they do not matter for the stability analysis) as $\dot{z} = Qz$. We may simply set the matrices of reaction coefficients equal to identity matrices, then with

$$B = \begin{bmatrix} 0 & -C' \\ 0 & 0 \end{bmatrix}$$

we can write our extended version (1″), (2′) as $\dot{z} = Qz + \gamma Bz$, which gives a Newton-like form

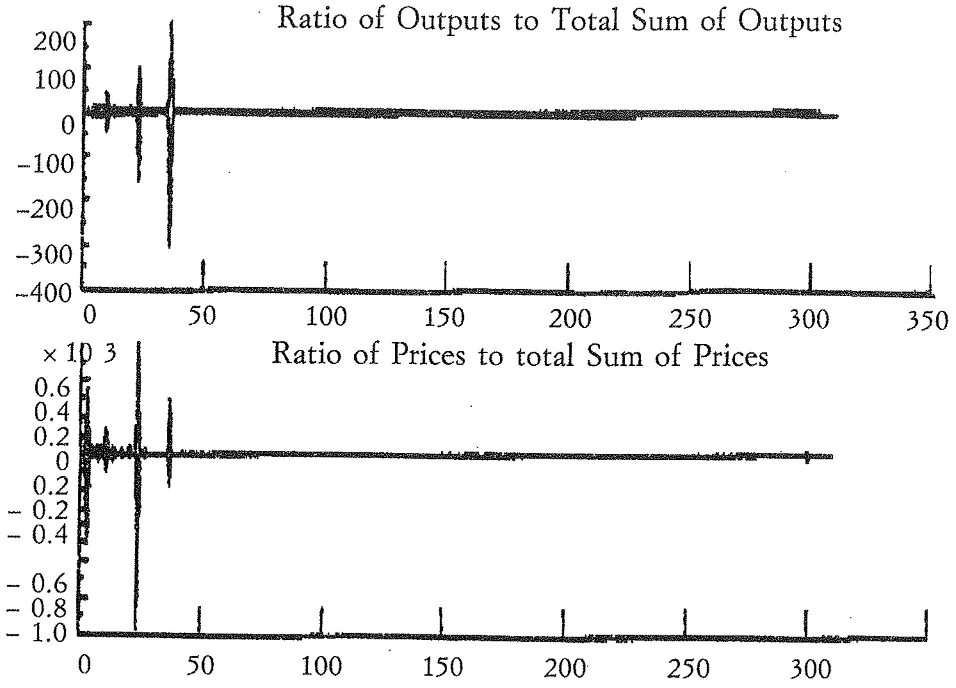

Figure 4

Simulation results for the same matrix as used in section IV.1, with $R = .9R^*_m$ and $G = .75R^*_m$, $\gamma_1 = 1, \gamma_2 = 1000$; and $\gamma = 5$, are depicted in Figure 4.

In the simulations, the trajectories for (1″), (2′) first moved more erratically, but tended to smooth out fairly soon and finally converged toward the steady state values p^*, x^*.

In eigenvalue computations we again employed $d_{ij} = I$, $d_{ij} = I$ and γ_1, γ_2 as reaction parameters for the dual and cross-dual dynamics, respectively, and γ for the derivative control term. The following results appeared

$$G < R < R^*$$

dim	γ_1	γ_2	γ	R^*_m	R	G	NMT	NMR^*_m	real part	ratio
6	1	10000	.02	1.35	$.9R^*_m$	$.75R^*_m$	1000	650	6	6/650
6	1	10000	.05	1.35	$.9R^*_m$	$.75R^*_m$	1000	670	0	0/670
6	1	10000	.1	1.35	$.9R^*_m$	$.75R^*_m$	1000	660	0	0/660
6	1	1000	.1	1.35	$.9R^*_m$	$.75R^*_m$	1000	680	0	0/680
6	1	1000	.2	1.35	$.9R^*_m$	$.75R^*_m$	5000	3300	0	0/3300

$\dot{z} = (I - \gamma B)^{-1} Qz$. For simple cases Flaschel (1989) has shown that a strong γ can always give rise to stability of the modified dynamic system. This technique of the proof of stability, however, does not carry over to our system (1″), (2′). This negative result follows from Saari and Simon (1978).

Table 4

The eigenvalue computations demonstrate that even very strong crossdual reaction parameters, for instance $\gamma_2 = 10000$, can always be offset by a small derivative control on the rate of return on capital. Derivative control of type (10) seem to be very effective in controlling unstable trajectories.

Case 2: Derivative Control for Excess Demand Functions

Subsequently, the derivative control for the excess demand function was tried. The following additional term was appended to (2′)

$$\gamma d_{21} C(g)(x_{t+h} - x_t)/h = \gamma d_{21} C(g)(d_{11} C(g) x_t - d_{12} C(r)'p'_t + q_1) \quad (11)$$

In terms of eigenvalue computations we obtained

$G < R < R^*$

dim	γ_1	γ_2	γ	R^*_m	R	G	NMT	NMR^*_m	real part	ratio
6	1	1000	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	370	370	370/370
6	1	10	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	360	360	360/360
6	100	10	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	380	0	0/380
6	1	1000	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	390	390	390/390
6	100	100	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	310	0	0/310
6	100	300	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	330	40	40/330
6	100	300	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	345	345	345/345

Table 5

For the derivative control of the excess demand function, stability does not arise as easily as for the profit rate: only if γ_1 is as large as γ_2 or larger stability arises.

Case 3: Derivative Control for Rate of Return and Excess Demand

The last case to be explored is when one allows for a total derivative control. In this case (10) is appended to (1′) and (11) to (2′).

The following results were ontained from the eigenvalue computations with random matrices and with γ_3, γ_4 the reaction coefficients for (10) and (11).

$G < R < R^*$

dim	γ_1	γ_2	γ_3	γ_4	R^*_m	R	G	NMT	NMR^*_m	real part	ratio
6	1	10	.5	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	350	0	0/350
6	1	10	50	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	340	340	340/340
6	1	100	50	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	345	345	345/345
6	1	100	5	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	352	352	352/352
6	1	100	.5	.5	1.35	$.9R^*_m$	$.75R^*_m$	500	355	355	355/355
6	1	1000	5	5	1.35	$.9R^*_m$	$.75R^*_m$	500	360	360	360/360
6	1	1000	.5	.1	1.35	$.9R^*_m$	$.75R^*_m$	500	356	0	0/356
6	1	1000	.5	.3	1.35	$.9R^*_m$	$.75R^*_m$	500	340	340	340/340

Table 6

As can be observed from Table 6, total derivative control generates stability if either the cross-dual dynamics is only weakly connected to the dual dynamics or the reaction coefficients for the derivative control of the excess demand function are small relative to reaction coefficients of the derivative control for the rate of return.

Case 4: Partial Derivative Control Selectively Applied

On the other hand, one might be interested whether bounded oscillations arise when a derivative control term such as (10), which is essentially stabilizing the dynamics, is applied only selectively, for example at outer boundaries when the trajectories rapidly depart from the equilibrium. We have used the aforementioned Euclidean vector norm to define regions when the control term (10) is to become operative in an otherwise unstable system.[27] In this case, of course, stability analysis by computing the eigenvalue of the system is no longer possible, The study of the local dynamics will not be conclusive, since the local dynamics will always be unstable. Simulations, demonstrating the global behavior of the system, are provided instead.

[27] A simple rule was used here. We assumed that if the Euclidean norm exceeded a certain number, we tried different runs with numbers between 6 and 12, then the control term (10) was applied.

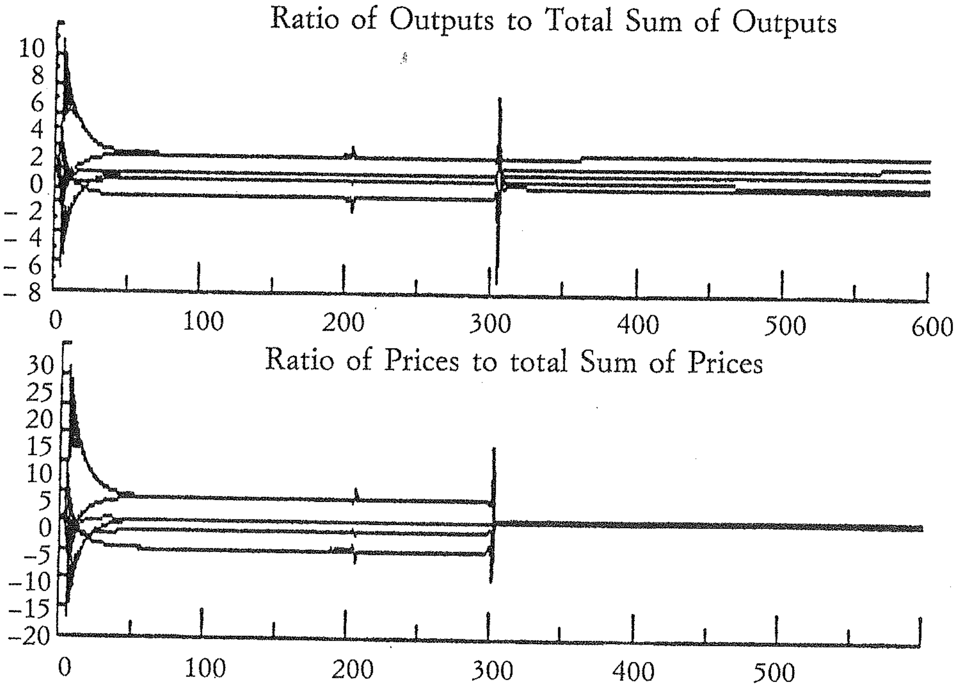

Figure 5

When the additional control term (10) is applied selectively at outer regions of the vector field,[28] as in our latter case, the trajectories remain bounded as depicted in Figure 5.

V. CONCLUSIONS

In this paper we have reconsidered and combined two formulations of microdynamic adjustment processes. We have discussed the stability properties of the cross-dual and dual adjustment processes, first isolated for each approach and then in composite form. In our view it adds realism to microdynamic studies if both approaches are synthesized and their dynamics studied jointly. In this way we may move closer to the study of empirically relevant adjustment processes as observed by many econometric studies (cf. Gordon, 1983, 1990; Taylor, 1980, 1986; and Semmler, 1984, ch. 3 for a summary of such studies).

We have applied a fairly new approach to the stability of large scale systems which seemed to be particularly well-suited for the type of composite system we had to analyze. The obtained results have been extended by means of computer studies. They suggest that stability will prevail even if a much more general parameter variety and certain nonlinearities are allowed for (in the above application of the decomposition-aggregation method). Yet, it is also shown that there are definite limits for such a generalizing conjecture. There exist a large class of systems which are easily made unstable in the linear case due to a strong influence of the off-diagonal coefficients.[29]

In that regard, the stabilizing aspects of derivative control terms for the overall structure in particular was investigated in more detail. In such situations the introduction of an additional force, for example stemming from the time rate of change of the rate of return, has been shown a convenient method for keeping the price and quantity fluctuations bounded. As demonstrated, the application of such an economically meaningful derivative control, globally or selectively applied, gave rise to stability (or at least bounded fluctuations). Derivative control on excess demand functions, in contrast with a derivative control on the rate of return, appeared to be less effective for stabilizing unstable trajectories. If both types of derivative control were applied to our system (1′), (2′), stability is not as easily obtained as for the derivative control of the rate of return.

With regard to the realism of such composite adjustment processes, one

[28] In future specifications the derivative control, instead of being applied in the above bang-bang fashion, might be utilized in a more continuous way.

[29] In Flaschel and Semmler (1989b), there are also systems reported, though economically not very meaningful ones, that are unstable independently of the reaction coefficients.

should express some caution since our modeled dynamics builds on a fairly simplified version of effective demand. Moreover, we have not analyzed the feedbacks resulting from supply constraints (labor, other inputs, or finance) to the output and price dynamics and the like. In addition, in studying the composite dynamics we worked with a (square) constant coefficients' matrix, which implies that process and product innovation is abstracted from.[30] However, the proposed synthesized dynamics of cross-dual and dual type appear to be interesting enough to encourage further extensions and generalizations in future research.

APPENDICES

Appendix I

In what follows, we want to briefly outline the decomposition-aggregation method of connective stability analysis following Siljak (1978), and then apply this method to our composite Keynesian-Classical system.

First, connective stability has to be defined. One-shot stability analysis attempts to prove stability for a dynamic system without going through an analysis of its basic component parts. In a composite system with interconnected basic subsystems, however, a composite type of stability analysis may be more appropriate than the well-known single step approaches to stability by means of single Liapunov functions. Here, one attempts to show that a dynamic interconnection, properly limited, will remain stable when stable, isolated subsystems are aggregated in various ways. Overall stability may arise, even in the case when some subsystems are unstable, but their influence on the overall dynamics is limited. This type of stability analysis is termed 'connective stability' in Siljak (1978).

Roughly speaking, connective stability means that the fully connected system (described by an interconnection matrix E) is stable as well as all of its structural perturbations which do not remove one of the self-contained subsystems from this structure, i.e., which contain at least the initially given decoupled substructure. Since in our case the Keynesian dynamics represents the self-contained and asymptotically stable subsystem, we consider this as the basic subsystem and the classical dynamics as a subsystem connected to the former. Switching on and off the classical dynamics can be regarded as a perturbation of the Keynesian dynamics. On the other hand, switching on (or off) Keynesian types of adjustments in the classical context demands an analysis of partially stable composite systems which was not attempted in this paper.

[30] For an extension of the cross-dual approach to the cases of process and product innovations and extinctions, cf. FLASCHEL and SEMMLER (1990).

Interesting methods for studying connective stability for composed systems by a decomposition-aggregation procedure are provided by the concepts of vector differential inequalities and vector Liapunov functions as elaborated in Siljak (1978, ch.2). In what follows we want to briefly outline this decomposition-aggregation method of connective stability analysis following Siljak, and then apply this method to our composite Keynesian-Classical system.

The connective stability of the equilibrium $z^* = 0$ of a system composed by connecting stable, initially isolated systems can be investigated in three steps.

First, one formulates an interconnected dynamical system from the knowledge of its basic componets, their internal dynamic structure and various conceivable interactions between these basic components. In general this may result in a complicated system. In our case (1), (2) this general approach can, however, be reduced to the following system:

$$\begin{aligned} S_1: \ \dot{z}_1 &= \bar{A}_{11} z_1 + e_{12} A_{12} z_2 \\ S_2: \ \dot{z}_2 &= \bar{A}_{22} z_2 + e_{21} A_{21} z_1 \end{aligned} \tag{A1}$$

In this system the matrices $\bar{A}_{11}$, $\bar{A}_{22}$ represent the independent Keynesian subsystems and e_{12}, e_{21} represent the elements of an interconnection matrix E, which, for symplicity, can be set to 1. The first part of (A1) thus represents the decoupled system which is not modified by the structural perturbations allowed for.[31]

Second, the asymptotic stability of each decoupled system in (A1): $\dot{z} = \bar{A}_{11} z_1$, $\dot{z} = \bar{A}_{22} z_2$ is assumed as given (or proved). As Liapunov functions for the isolated subsystems A_{11}, A_{22} we can then take

$$v_1(z_1) = (z_1' H_1 z_1)^{1/2}, \ v_2(z_2) = (z_2' H_2 z_2)^{1/2} \tag{A2}$$

where the positive definite and symmetric matrices H_1, H_2 are determined by

$$\bar{A}_{11}' H_1 + H_1 \bar{A}_{11} = -I, \qquad \bar{A}_{22}' H_2 + H_2 \bar{A}_{22} = -I \tag{A3}$$

The total time derivatives of (A2) are [see (A3), ($i = 1,2$)]:

$$\begin{aligned} \dot{v}_i &= (\text{grad } v_i)' \dot{z}_i = (\text{grad } v_i)' \bar{A}_{ij} z_i \\ &= (v_i^{-1} H_i z_i)' \bar{A}_{ij} z_i = -(1/2) v_i^{-1} (z_i' z_i). \end{aligned} \tag{A4}$$

From (A2), (A3) and (A4) estimates for these Liapunov functions are then produced as follows ($i = 1,2$; note the minus sign in (A4)):

$$\theta_{i1} \| z_i \| \le v_i \le \theta_{i2} \| z_i \|, \ \dot{v}_i \le -\theta_{i3} \| z_i \|, \ \| \text{grad } v_i \| \le \theta_{i4}, \tag{A5}$$

[31] Note here that SILJAK (1978: 33) uses a different notation to represent this case and that he in general allows for further feedbacks of z_i on z_i which may be switched on and off through e_{ij} and structural perturbations.

with the following positive scalars θ_{ij}

$$\theta_{i1} = \Omega_m^{1/2}(H_i) , \qquad \theta_{i2} = \Omega_M^{1/2}(H_i) ,$$
$$\theta_{i3} = \frac{1}{2\Omega_M^{1/2}(H_i)} , \qquad \theta_{i4} = \frac{\Omega_M(H_i)}{\Omega_m^{1/2}(H_i)} .$$

Here Ω_m and Ω_m denote the minimum and maximum eigenvalues of the symmetric and positive definite matrices H_1, H_2.

Third, the functions v_1, v_2 are representatives of the stability of each subsystem $\bar{A}_{11}$, $\bar{A}_{22}$, and we can now study the stability of the aggregate system S composed of S_1, S_2 by considering appropriate compositions of these two stability indicators, no longer considering the dynamic interaction within each subsystem in its details. The total time derivative along the solutions curves of each interconnected subsystem S_i of (A1) is ($i, j = 1,2$):

$$\begin{aligned} \dot{v}_i &= (\text{grad } v_i)'\dot{z}_i \\ &= (\text{grad } v_i)'[\bar{A}_{ii}z_i + e_{ij}A_{ij}z_j] = \dot{v}_i(A4) + (\text{grad } v_i)'e_{ij}A_{ij}z_j \end{aligned} \tag{A6}$$

where $\dot{v}_i$ (A4) is given by (A4). This gives rise to

$$\dot{v}_i \leq -\theta_{i3}||z_i|| + e_{ij}||\text{grad } v_i|| \; ||A_{ij}z_j|| \tag{A7}$$

which together with the constraint on the nonsymmetric interaction matrix A_{ij}

$$||A_{ij}z_j|| \leq \epsilon_{ij}||z_{ij}||, \; \epsilon_{ij} = \Omega_M^{1/2}(A_{ij}'A_{ij}) \tag{A8}$$

finally gives (because of the minus sign in (A7)):

$$\begin{aligned} \dot{v}_1 &\leq -\theta_{12}^{-1}\theta_{13}v_1 + e_{12}\epsilon_{12}\theta_{14}\theta_{21}^{-1}v_2, \\ \dot{v}_2 &\leq e_{21}\epsilon_{21}\theta_{24}\theta_{11}^{-1}v_1 - \theta_{22}^{-1}\theta_{23}v_2. \end{aligned} \tag{A9}$$

This system can be rewritten by means of the vector Liapunov function $v = (v_1, v_2)'$ as one vector inequality

$$\dot{v} \leq Wv. \tag{A10}$$

where the aggregation matrix W is defined by ($i,j = 1,2$):

$$w_{ij} = \begin{cases} -\theta_{i2}^{-1}\;\theta_{i3} , & i = j, \\ e_{ij}\;\epsilon_{ij}\;\theta_{j1}^{-1}\;\theta_{i4} , & i \neq j, \end{cases} \tag{A11}$$

The connective stability of the overall system then follows from a stability proof for the aggregated system (A10). In order to prove this result, Siljak (1978, ch.2) introduces the comparison principle for vector differential inequalities by majorizing the function v appropriately. This principle uses for comparison the differential equation

$$\dot{r} = \bar{W}r$$

with the initial condition $v_o = r_o$, and $\overline{W}$ the aggregation matrix corresponding to the fundamental interaction matrix E

$$\overline{w}_{i,j} = \begin{cases} -\theta_{i2}^{\;-1}\,\theta_{i3}\,, & i = j, \\ \theta_{j1}^{\;-1}\,\theta_{i4}\,, & i \neq j, \end{cases} \tag{A12}$$

If the matrix $\overline{W}$ is stable and if we know (for all of our interconnections)

$$v(t) \leq r(t), \qquad t \geq t_o,$$

then one can conclude that $\lim_{t->\infty} v(t) = 0$ holds true for all such E.

We thus obtain connective stability, as defined above, for the whole system S.[32]

The above type of decomposition-aggregation analysis by means of vector Liapunov functions consequently gives rise to the following

Theorem: Given that (1) asymptotic stability of each decoupled subsystem is established and described by the estimates (A5) obtained for the Liapunov functions v_1 and v_2, (2) the constraints (A8) on the interactions $A_{12}z_2$ and $A_{21}z_1$ between the subsystems S_1 and S_2 hold, and (3) stability of the aggregate matrix $\overline{W}$ corresponding to the fundamental interconnection matrix $\mathfrak{z}$ has been proved, then the system S is stable for all interconnection matrices E, that is, it is connectively stable.

A variety of additional related stability concepts and theorems other than the above are investigated in Siljak (1978) and Michel and Miller (1977). The interested reader may consult this literature and the more detailed application of theorems to be found there to the classical-Keynesian composite dynamics in Flaschel and Semmler (1989b).

Two important final remarks may be added concerning extensions of the above theorem.

Remark 1: For a dynamically reliable large-scale system one would expect that the system is allowed to disintegrate itself and then to reintegrate itself during its functioning. The above discussed class of structural perturbations can be generalized into this direction by means of time-dependent interconnection matrices $E(t)$ to allow for on-off participations of subsystems in the course of time (cf. Siljak, 1978).

Remark 2: In even more general terms, it is also not necessary that all connected subsystems are stable when isolated. Unstable subsystems may be permitted to be parts of a large composite system provided, of course, that sufficiently strong stabilizing cross-feedbacks are present at all time. When interconnection matrices are carefully chosen, unstable subsystems[33]

[32] Cf. Siljak (1978:37ff.) for further details.

[33] Thus, in the context of our system above the price dynamics sketched in (3) — or, in general, a price dynamics stemming from an excess demand function — can be allowed to be unstable and its instability still might be turned into stability through the interconnection with other stable subsystems.

can be allowed for and the system may nevertheless exhibit connective stability, cf. Siljak (1978, ch. 2.6).

Appendix II. Search for Unstable Matrices through Eigenvalue Computations

1. The eigenvalue computations were undertaken with randomly generated matrices. As reported in section IV several size classes of random matrices were explored. To illustrate the type of random matrices employed and to exemplify how a matrix can become unstable through a variation of the off-diagonal reaction coefficients, we take a 6 dimensional random matrix A and compute the eigenvalues for a matrix Q (which depend on the reaction coefficients). A matrix A being generated was for example:

$$A = \begin{bmatrix} .0776 & .1453 & .1439 & .06978 & .1426 & .1073 \\ .03498 & .1225 & .1441 & .1246 & .1558 & .1321 \\ .1017 & .09612 & .03554 & .05683 & .04618 & .1454 \\ .1215 & .1347 & .1167 & .1616 & .1013 & .04532 \\ .1363 & .02503 & .04135 & .02869 & .05348 & .002448 \\ .1083 & .05687 & .09887 & .1390 & .08112 & .1479 \end{bmatrix}$$

The A matrix above has a $R^* = 1.7224$. The corresponding matrix Q is

$$Q = \begin{bmatrix} \gamma_1 d_{11} C(g) & -\gamma_2 d_{12} C(r)' \\ \gamma_2 d_{21} C(g) & \gamma_1 d_{22} C(r)' \end{bmatrix}$$

For $G = 1.1$ and $R = 1.6$ and with $\gamma_1 = 1$, $\gamma_2 = 1000$ we get the following eigenvalues for the matrices Q_k, Q_{cl} and Q.

Eigenvalues of

Q_k		$Q_{c,l}$		Q	
Real part	Im. Part	Real Part	Im. Part	Real Part	Im. Part
− 0.361	0	1.42 − 13	157.5	− 0.209	157.5
− 1.086	0	1.42 − 13	− 157.5	− 0.209	− 157.5
− 1.012	0.07	− 4.2 − 14	1158.5	− 1.146	1158.5
− 1.012	− 0.07	− 4.2 − 14	− 1158.5	− 1.146	− 1158.5
− 0.934	0.04	− 1.59 − 14	1059.7	− 1.055	1059.7
− 0.934	− 0.04	− 1.59 − 14	− 1059.7	− 1.055	− 1059.7
− 0.07	0	− 1.77 − 15	993.4	− 0.98	993.4
− 1.126	0	− 1.77 − 15	− 993.4	− 0.98	− 993.4
− 1.017	0.1	− 6.35	901.6	− 7.24	901.5
− 1.017	− 0.1	− 6.35	− 901.6	− 7.24	− 901.5
− 0.904	0.05	6.35	901.6	5.45	901.5
− 0.904	− 0.05	6.35	− 901.6	5.45	− 901.5

Table A1

Note that Table A1 shows that in a case when asymmetry of the matrices for Q_{cl} (and Q_k) is lost, the cross-dual dynamics can become unstable ($G \neq R$). With scalars of reaction coefficients $\gamma_1 = 1$, $\gamma_2 = 1000$, the corresponding composite Q matrix has a pair of real parts of eigenvalues $\lambda_{max}(A) = 5.45$, indicating that the composite matrix Q is unstable. A decrease of γ_2 to 100 resulted in no positive real parts of the eigenvalues. The composite matrix Q became stable. A further decrease of γ_2 to 1 kept the composite matrix Q stable. The matrix A above with $\gamma_2 = 1000$ for the off-diagonal terms in the appropriate Q matrix was used for the simulations in sect. IV, Figure 4, above.[34]

2. Types of counterexamples as the one above have motivated us to introduce a derivative control term, since the negative feedback of the dual system is in general insufficient to turn the center-type stability of the cross-dual system (or its instability as in the case above) into stable trajectories of the composite system. Derivative control as discussed in section IV.2 above can again turn an unstable matrix Q into a stable one.

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[34] There also exist systems where instability prevails somewhat independently of reaction coefficients (FLASCHEL and SEMMLER 1989b, Appendix). Of course, even though, as in this case, instability may prevail, there always exist reaction coefficients, as also demonstrated in FLASCHEL and SEMMLER (1989b), that render the system stable (since our composite system is HICKSIAN).

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