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Part III

Gravitation Processes and Technical Change: Convergence to Fractal Patterns and Path Stability

Ulrich Krause*

I. INTRODUCTION: SETTING THE PROBLEM

"We must consider stability of motion rather than of a point". (M. Morishima in a discussion remark at the Siena Conference 1990).

The classical economists developed the idea of prices gravitating towards so-called prices of production as center of gravity. (Cf. [8], [20]). Cum grano salis, the classical idea in its original verbal formulation looks rather simple and not implausible. However, the attempts of numerous authors, to state the classical idea of gravitation in precise terms and to argue carefully for its validity, show a different picture. Today, there is a supply of quite different models, some refuting the classical idea, some confirming it, though by using additional hypotheses, and some doing something in between. Moreover, along with the modeling of the classical idea of gravitation, a whole bunch of technical problems, a few of them nice but many of them nasty, began to throw the original charming little idea into the shade. As disappointing as this may be, what I consider to be a positive byproduct is a widespread awareness concerning the issue of dynamics.

In the following pages I would like to deal with a topic which I consider an important one for dynamical processes in general and which shows up in particular for gravitation processes if technical change is admitted. To my knowledge, technical change, and even the choice of techniques, has been left out of consideration in almost all discussions of gravitation processes. (Choice of techniques is considered in [5,6], [13], and in connection with technical change in [10]). This I consider a major drawback for several reasons, one being that technical change should not be excluded from models where time approaches infinity.

The idea of a *gravitation process* may be, without specifying technical details, modeled as a dynamical process by which all the entities considered are attracted to one single center of gravity. Since in economics it has proven to be useful to model dynamical systems in discrete time, in what follows I shall consider *discrete dynamical systems*. Such a system in its simplest form is given by some set X of entities (like prices or price-quantity pairs as in cross-dual models) together with a selfmapping of X ("the law of motion"). The motion starting from a point x_0 of X is described by the *path* (orbit) $\{x_0, f(x_0), f^2(x_0), \dots\}$ where f^t denotes the *t-th* iterate of f. If x^* is a center of gravity for the dynamical system (X,f), then $f^t(x_0)$ should approach x^* for any x_0 and t approaching infinity. But why should there be anything like a gravitation center? An appealing intuitive idea, which is often implicit in verbal descriptions, is that of a law of motion f which brings any two points closer to each other. In fact, for such a *contraction* f the gravitating behavior asked for takes place. (By Banach's contraction mapping principle, provided some technical requirements are met). Now, the question I will address, is, what happens if the law of motion f itself changes in time? An example of the latter is the change of techniques during the process of gravitation, because the mapping f (the difference equation, the differential equation, or systems of these) depends on the technology. Such a change in f may be caused also by other circumstances. *E.g.*, for a fixed but nonlinear technology (*i.e.* unit costs do depend on the level of output) a change in the output may take place and modify the law of motion (cf. [14)]. The answer to the above question of what will happen depends, of course, on the properties of the particular system under consideration. A special kind of technical change which does not destroy the gravitational process is examined in [10]. However, this is not what one would expect in general. Quite the opposite will be the case in general, namely the feature of gravitation disappears and there will be no convergence to long-period positions. This is due to the several different techniques which act as several gravity centers. The whole situation reminds one of Buridan's ass, and the resulting dynamics looks like fluttering around between several goals. A new phenomenon arises in that the dynamical system, instead of approaching a single point, approaches a so-called *fractal set*, which may be quite complicated. Nevertheless, a stability property called *path stability* still holds, meaning that any two paths finally come arbitrarily close to each other. This may also be viewed as a stability property of the motion itself, since any path, when perturbed, comes back finally to its original motion.

What has been said in broad terms in this introduction is explained in more detail in section 2 by means of a simple concrete example. Section 3 makes precise the idea of path stability in general and presents a condition when this property holds. Section 4 reports briefly on path stability for prices in Leontief models, even nonlinear ones, and has a look at the theory of *positive discrete dynamical systems* which is behind it.

2. A SIMPLE EXAMPLE OF TECHNICAL CHANGE

The example I shall discuss is extremely simple and highly stylized and it is presented only for illustrative purposes. The simplest case will be considered, that is, there will be one commodity only called 'corn' which serves as input and output of production as well as consumption good. There will be two techniques only, which are used to produce corn in two different seasons, say 'summer' and 'winter'. The change of techniques simply consists then in switching from one technique to the other following the natural order of the two seasons. More precisely, suppose the 'summer-technique', s-techinque for short, uses $\frac{1}{3}$ units of corn but no labor to produce 1 unit of corn. (The remaining $\frac{2}{3}$ units may be used for consumption). As corn growing is more difficult in the winter, the corresponding w-technique uses, besides the $\frac{1}{3}$ units corn for sowing in addition $\frac{2}{3}$ units of labor. (Although the numbers are completely fictitious, there is some reason for choosing precisely these numbers, as will be seen later on.) The dynamics I am interested in, is that of prices, for which it is assumed that the corn price in period $t + 1$ equals the cost of production in period t. (The implicit assumption of no profits is made only to keep things simple. For the case of profits see section 4.) This idea of prices driven by costs is also part of classical economics (cf. [1]). If p_t denotes the price of corn per unit in period t and the wage rate is set 1 without loss of generality, then the dynamics determined by the w-technique alone is given by

$$p_{t+1} = f(p_t) = \frac{1}{3} p_t + \frac{2}{3} \cdot$$

Assume for a moment that only the w-technique is applied and there is no technical change. Then a simple calculation yields

$$P_{t+1} = f^t(p_1) = \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3} + \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} + \ldots + \left(\frac{1}{3}\right)^{t-1} \cdot \frac{2}{3} + \left(\frac{1}{3}\right)^t p_1$$

Since $\left(\frac{1}{3}\right)^t p_1$ converges to 0 for p_1 arbitrary if t approaches infinity, one obtains by using the formula for the geometric series that

$$p^* = \lim_{t \to \infty} p_{t+1} = \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 1.$$

That is, corn prices gravitate, irrespective of the initial price p_1, to the center of gravity $p^* = 1$, which in fact is also the unique equilibrium for the production price equation $p = \frac{1}{3}p + \frac{2}{3}$. Similarly, if only the s-technology is applied, the dynamics is given by $p_{t+1} = g(p_t) = \frac{1}{3}p_t$, and corn prices tend to the center of gravity $p^{**} = 0$, which is the unique equilibrium for the production price equation $p = \frac{1}{3}p$. The crucial point now is that in considering technical change we have to consider both techniques together, one alternating with the other, beginning with, say, the s-technique. Hence the dynamics for (this kind of) technical change is given by so-called *inhomogeneous iteration*... $f \circ g \circ f \circ g(p_1)$, where the expression has to be read from right to left and where "$\circ$" denotes composition of mappings. A simple calculation yields the more precise description

$$p_{t+1} = \begin{cases} (f \circ g)^{t/2}(p_1) & \text{for } t \text{ even} \\ g \circ (f \circ g)^{(t-1)/2}(p_1) & \text{for } t \text{ odd} \end{cases} \qquad (*).$$

One might say that there are two centers of gravity, 0 and 1 respectively, to which the corn price is attracted alternately due to technical change. Will the corn price gravitate? Maybe converge to some average, maybe to $\frac{1}{2}$? The answer is 'no' which can be seen as follows without knowing the dynamic behavior in detail. Consider the composite law $h = f \circ g$ which is given by $h(p) = \frac{1}{9}p + \frac{2}{3}$. Similarly as above for f, one calculates

$$p_{2n+1} = h^n(p_1) = \frac{2}{3} + \frac{1}{9} \cdot \frac{2}{3} + \ldots + \left(\frac{1}{9}\right)^{n-1} \frac{2}{3} + \left(\frac{1}{9}\right)^n p_1$$

to find $\lim_{n \to \infty} p_{2n+1} = \frac{3}{4}$.

From the description (*) one gets for odd $t = 2n + 1$

$$\lim\ p_{2n+2} = \lim_{n \to \infty} g \circ (f \circ g)^n(p_1) = g\left(\frac{3}{4}\right) = \frac{1}{4}.$$

Because of $\frac{1}{4} \neq \frac{3}{4}$ the conclusion must be that the sequence of prices does not converge in the case of technical change. Whatever the price path may be, it can be said that the path will be finally independent of the initial price p_1. Namely, let p_1 and q_1 be two different initial prices and let p_t and q_t be the corresponding prices in period t as determined by equation (*). Then $p_{2n+1} - q_{2n+1}$ tends to $\frac{3}{4} - \frac{3}{4} = 0$ and $p_{2n} - q_{2n}$ tends to $\frac{1}{4} - \frac{1}{4} = 0$.

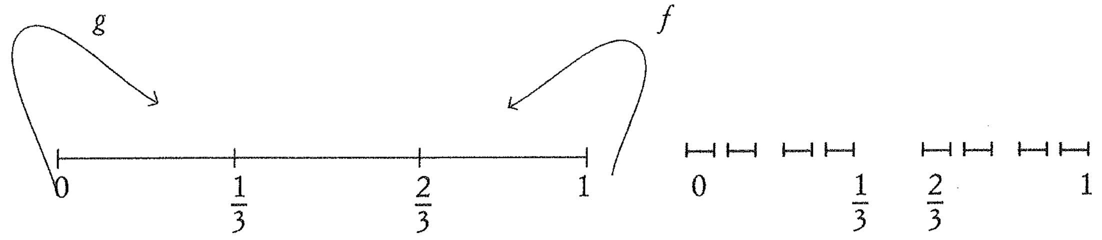

Figure 1 - Cantor set (stage 3)

Hence $p_t - q_t$ tends to 0 for $t \to \infty$, meaning that two arbitrary paths finally become arbitrarily close to each other. This property is called *path stability*. (In the next section it is shown that this property prevails in a rather general setting.) It can be seen, by direct calculation or by general arguments as in section 3, that path stability does hold also for the following kind of technical change in the simple example under consideration. Instead of applying them strictly alternately, the two techniques may follow each other in some arbitrary but fixed manner like... $g \circ f \circ g \circ g \circ g \circ f(p)$.

But what can be said about the detailed behavior of a typical path? Obviously, the mappings $f(p) = \frac{1}{3}p + \frac{2}{3}$ and $g(p) = \frac{1}{3}p$ map the unit interval [0,1] into itself. Hence the dynamic system (*) stays within [0,1]. In particular, there is at least one path contained completely in [0,1]. By path stability it follows that for any starting point $p_1 \geq 0$ the path stays finally in [0,1]. It is even true that g maps the interval [0,1] into the interval $\left[0, \frac{1}{3}\right]$ and that f maps [0,1] into $\left[\frac{2}{3}, 1\right]$. If one uses this fact at every stage of the inhomogeneous iteration, it is easy to see that every path is attracted by the so-called *Cantor set* as is indicated in the following figure.

The Cantor set is obtained as follows, from the interval [0,1]. First, the interval [0,1] is divided into three intervals of equal length and the (open) middle-interval is removed. Then each of the remaining intervals $\left[0, \frac{1}{3}\right]$, $\left[\frac{2}{3}, 1\right]$ is again divided into three intervals of equal length and the (open) middle-intervals are removed. Etc. In the limit the Cantor set is obtained which is the simplest of the *classical fractals*. (For fractals cf. [2], [17].) That the Cantor set is an attractor for the dynamic system under consideration may be expressed also by saying that every path must finally be contained in any given set obtained at some stage of the Cantor set construction. To summarize, it has been shown by means of a simple example that, in the case of technical change, prices do not gravitate towards a single price but towards a price pattern which is contained in some fractal set. For any given type of technical change this price pattern may be simple and represent only a small part of the Cantor set. For strictly alternating techniques, *e.g.*, the price pattern consists of the two points $\frac{1}{4}$ and $\frac{3}{4}$. Depending on the type of technical change, the limiting price pattern may however be any subset of the Cantor set. In case the two techniques change in a purely random manner, then, in fact, the limiting price pattern fills the whole Cantor set.

In a manner similar to that for the extremely simple corn example one may also analyze the multi-commodity case as it is studied in Leontief or Sraffa models. Since fractals are geometrically interesting only from two dimensions onwards, I shall briefly comment on what will happen for two commodities. Consider three techniques $\left(A^{(i)}, l^{(i)}\right)$ for $i = 1, 2, 3$, where $A^{(i)}$ is the 2×2 input-output matrix and $l^{(i)}$ is the 2-vector of labor inputs. Suppose these techniques are simply as follows:

$$A^{(1)} = A^{(2)} = A^{(3)} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad l^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad l^{(2)} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad l^{(3)} = \begin{bmatrix} 0.25 \\ 0.4331 \end{bmatrix}.$$

Let f_1, f_2, f_3 be the cost functions induced, with wage rate equals 1, and suppose that the price of the next period is given by the costs of this period. Again, taking only one single technique (A, l) into account, there exists a unique solution p^* to the Sraffian system $p = pA + l$ of production prices, and prices set by producers do converge for all initial prices to p^*. The picture becomes drastically different if change takes place between the three techniques. It turns out that, whatever the particular way of change between the techniques, the limiting price pattern must be part of the so-called *Sierpinski triangle* as indicated in the following figure (cf. [2]).

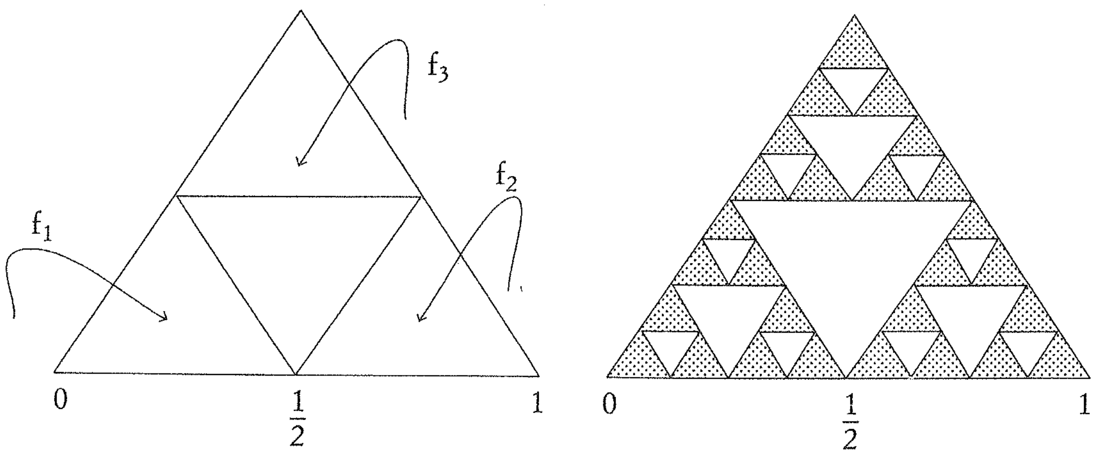

Figure 2 - Sierpinski triangle (stage 3)

Again, from the general principle shown in the next section it follows that path stability still holds there.

For illustrative purposes the numbers in the above examples were choosen in such a way as to arrive at the simple classical fractals shown. Using other numbers, other, and partly more complicated, fractals would result. (On the fantastic world of fractals see [17].) The process referred to of getting fractals by randomly playing a finite number of affine mappings is also called the *chaos game*. (For a detailed analysis of the game see [2].) Of course, to treat technical change properly one has to admit an infinite sequence of techniques, which induces an infinite sequence of cost functions f_1, f_2, f_3, ..., f_n being the cost functions in period n. (In continuous models of technical change the case is even worse, since technology depends continuously on the time variable t.) The detailed behavior of paths in this case has not yet been explored. It will follow, however, from the next section that here too path stability does hold, provided a certain condition is met. (Cf. also [11].) This issue as well as the question of what will happen if profit is no longer zero or if the technology is not longer linear will be picked up in section 4.

3. THE GENERAL IDEA OF PATH STABILITY

To get a clear concept of path stability as well as to capture more difficult situations (as in the next section) it is worthwhile to put certain aspects discussed in the previous section into some more abstract framework.

Let X be a (nonempty) set equipped with a metric d for measuring distances on X. Assume that the *metric space* (X,d) is complete. Examples from the previous section are $X = \mathrm{R}_+$ or $\mathrm{R}^2{}_+$ and d given by the absolute value and the Euclidean metric, respectively. (Thereby $\mathrm{R}^n{}_+ = \{(x_1, \dots x_n) \mid x_i \in \mathrm{R},\ x_i \geq 0 \text{ for } i = 1, \dots, n\}$.) Let f be a selfmapping of X which is a *contraction* for d, *i.e.* there exists some *factor of contraction* $0 \leq c < 1$ such that $d(f(x), f(y)) \leq cd(x,y)$ for all x, $y \in X$. Examples from the previous section are the mappings $f(x) = \frac{1}{3}x + \frac{2}{3}$, $g(x) = \frac{1}{3}x$ on R_+ and the mappings $f_i(x) = A^{(i)}x + l^{(i)}$ for $i = 1,2,3$ on $\mathrm{R}^2{}_+$ (with metrics as mentioned before). As has been pointed out in section 1, a contraction may serve very well as a model for a gravitation process. Indeed, Banach's contraction mapping principle yields that for a contraction f there exists a unique equilibrium x^*, *i.e.* $f(x^*) = x^*$, and for any starting point x in X the path $\{f^n(x) \mid n \geq 0\}$ gravitates to x_0. This harmonic picture is destroyed if, instead of iterating one and the same f, (so-called *homogeneous iteration*), several different contractions are used in the process of iteration (so-called *inhomogeneous iteration*). The multiplicity of contractions brings about a multiplicity of centers of gravity which may create a very irregular motion.

(But, fortunately, there will be no gravitation between the centers of gravity themselves.) Nevertheless, path stability holds, as will be seen in a moment. Let F be an arbitrary family of contractions on X and let c_f be the contraction factor of f in F. Consider a mapping $M : \mathrm{N} \rightarrow F$ which specifies for every natural number n a function $M(n)$ within F. An *M-path* on X is defined recursively by $x_{n+1} = M(n)(x_n)$ together with a starting point x_1 in X. The system (F,M) is said to possess *path stability* whenever, for any two M-paths, it holds that the distance $d(x_n, y_n)$ converges to 0 for $n \rightarrow \infty$.

Lemma If $\sup \{c_f | f \in F\} < 1$, then path stability holds for every specification M.

Proof By the definition of contraction factors, it follows with $c = \sup \{c_f | f \in F\}$ that $d(f(x), f(y)) \leq c\, d(x,y)$ for all x, y in X and all f in F. Using induction on $n \geq 0$ one obtains for any two M-paths starting from x_1 and y_1 respectively

$$d(x_{n+1}, y_{n+1}) \leq c^n d(x_1, y_1) \tag{*}$$

For, inequality (*) is true for $n = 0$. If (*) holds for n, then it follows that $d(x_{n+2}, y_{n+2}) = d(M(n+1)(x_{n+1}), M(n+1)(y_{n+1})) \leq c\,d(x_{n+1}, y_{n+1}) \leq c \cdot c^n d(x_1, y_1) = c^{n+1} d(x_1, y_1)$ by using the induction hypothesis. This shows (*). The lemma then follows by letting $n \rightarrow \infty$ in (*).

A system $x_{n+1} = M(n)(x_n)$ as in the lemma one encounters in the case of technical change. The set F represents thereby the, possibly infinite, set of techniques, or rather, the cost functions for these techniques. $M(n)$ is the technique chosen in period n. According to the lemma, path stability prevails if the cost functions are contractions for some suitable metric and the contraction factors are all bounded from above by a constant less than 1. The latter assumption is satisfied, of course, if the family F is finite. Just how useful this little lemma can be may be seen by looking back at the calculations made for the example in the previous section. As is shown by this example, path stability for inhomogeneous iteration does not necessarily imply convergence to a single point. Path stability may b viewed as a kind of stability of motion in the following sense. If the particular kind of motion exhibited by a path is disturbed from the outside, then, after a while, the disturbed path must show the same kind of motion as the original path.

4. PRICE DYNAMICS IN LEONTIEF MODELS

In this section, I shall sketch briefly how one may obtain path stability even in nonlinear Leontief models. (For a detailed analysis, see [14].) The examples discussed in section 2 belong to the realm of Leontief or Sraffa models. The first and outstanding contribution concerning the problem of price dynamics in a linear Leontief model was made in [15]. Further contributions comprise [3], [5,6], [7], [10], [13]. Things become much more difficult in nonlinear Leontief models where the input-output coefficients are admitted to depend on the quantities of the goods produced. Fortunately, valuable work has already been done on nonlinear Leontief models, in particular in [15], [16], [19]. Actually, in [19] some kind of nonlinear Perron-Frobenius theorem has been developed, including the stability part, which then has been generalized in [15] and [16] and which has been proven to be very useful. More recent contributions are [4], [9], [12], [18]. However, all the work quoted on nonlinear Leontief models is not concerned with the dynamics of prices, but with the dynamics of quantities only.

The underlying philosophy for the nonlinear case is the same as for the examples discussed in section 2, viz. the classical credo of prices driven by costs ([1]). In a nonlinear framework, however, the minimum cost to produce y units of a commodity does depend on the quantity y as well as on prices and the wage rate. Hence there are several possibilities, to choose the relevant cost that drives the price of a particular commodity. Examples of relevant cost are marginal cost and average cost, respectively. Leaving special problems in treating the wage rate aside (cf. the different approaches in [5,6] and [14]), the relevant cost functions in a fixed period may be put together to obtain the cost operator T, which is a nonlinear selfmapping on $R^n{}_+$. Thereby, n is the number of goods and the *i-th* component $(Tp)_i$ of Tp is given by the relevant costs of producing a given amount of good i in the fixed period. In the linear case, e.g., and without choice of techniques, one would have $Tp = pA + l$, if $w = 1$ is given, and $Tp = p\bar{A}$, if $w = pc$ for some fixed consumption bundle c ($\bar{A}$ the augmented input-output matrix). Admitting profits, however, one cannot simply take T to be a function as in the previous section to define the dynamics. Also, the mapping T need not be a contraction for some metric. If, e.g., in the linear case $Tp = p\bar{A}$, then iterates of $\bar{A}$ need not be convergent. For these reasons one proceeds as follows. Prices are normalized by, say, $||p|| = \sum_{i=1}^{n} p_i$. The relevant space of prices then is $X = \{ p \in R^n_+ | \ \ || p || = 1 \}$. More important, the mapping T is also scaled down as $f(p) = \frac{Tp}{||Tp||}$ · f is a selfmapping of X and turns out to be the appropriate kind of function in the sense of the previous section. A suitable metric on X is given by *Hilbert's projective metric*, with respect to which f becomes a contraction, provided T meets some plausible properties of positivity (which are needed also in the linear case). By the way, even in the linear case the scaled down version f of T in general is not linear. Whereas in the linear case one uses Perron-Frobenius theory, some nonlinear version of this theory is needed in the nonlinear case. More generally the tools needed are from the very recent branch of *positive discrete dynamical systems*. Using these tools (cf. [11]), one may obtain path stability.

In the case of technical change, instead of a single cost operator T, one has to consider a sequence T_t, $t = 0, 1, \ldots$ of these operators. To the sequence of the corresponding normalizations f_t the lemma of the preceding section may be applied. The path stability obtained is an interesting property even in case of a linear technology. Another interesting case is that of output-dependent price dynamics for the nonlinear model. The problem in this case is that, even if the techniques do not change, the cost operator T will change with a change in output. As in the case of technical change, this destroys the process of gravitation, because of different centers of gravity belonging to different outputs. Nevertheless, it can be shown, in a manner formally very similar to that in the case of technical change, that path stability still holds.

5. CONCLUSION

The good old idea of a gravitation process needs to be formulated anew in the presence of technical change. More generally, a new formulation of this kind is needed in all cases where relevant circumstances change in time, as for example outputs in case of a nonlinear technology. The new formulation presented in this paper is by inhomogeneous iteration within the field of positive discrete dynamical systems. Broadly speaking, this method allows one to handle the phenomenon that the center of gravity itself does change during the process of gravitation. By means of simple examples it has been shown that for inhomogeneous iteration convergence to a single point can no longer be expected but, instead, convergence to some fractal pattern. Despite the possible irregularity of each single path, however, path stability still holds. This means that any two paths come finally arbitrarily close to each other. This may be viewed also as a stability of motion, in that any path, when perturbed from the outside, after a while shows the same motion as before. Furthermore it has been shown that path stability is quite a universal feature in the realm of gravitation processes, in the sense that any process which is obtained by mixing (finitely many) processes of gravitation — the result of which is not a process of gravitation in general — does show path stability.

FB Mathematik und Informatik, Universität Bremen.

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