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Part III

# The Dynamic Stability of Production Prices: A Synthetic Discussion of Models and Results*

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## 1. INTRODUCTION

In the works of the Classical economists and Marx the relationship between actual prices and production prices had been described in terms of a "gravitation" process: the movements of capital from low-profit sectors to high-profit sectors would guarantee that the market (*i.e.* actual) prices would always "gravitate around" production prices.

In recent years that relationship has been reconsidered and investigated by means of mathematical models and the theory of dynamic systems.

Here, we shall discuss these models, their results and the implications of these results. These models fall into two main categories: 1) models of cross-dual dynamics; 2) models of full-cost.

In the *full-cost* models, the price of each commodity is simply determined by its production cost — measured at the normal level of utilization of capacity — plus a target rate of return.

In the *cross-dual* models, the price of each commodity depends on excess-demand, hence on current and planned outputs; the growth of output in each sector, in turn, depends on the rates of profit, hence, on prices.

The latter feature of cross-dual models can be considered as roughly equivalent to the Classical idea of capital moving from low — to high-profit sectors. Much more disputable is the connection between their former feature — price formation based on excess-demand — and the Classical theory of market prices.

As to the full-cost models, a relationship between them and the Classical "gravitation" process seems difficult to find.

Thus, although the Classical theory of gravitation is the main inspiration for the models we shall be concerned with, they cannot be considered simply as a modern and formalized version of it.

In these models, a vector of production prices (if conceived as *relative* prices) is usually an equilibrium vector or part of it, where by equilibrium

* This is a simplified and abridged version of the paper presented at the Conference.

we mean — following the theory of dynamic systems — a point in the space of the state variables that, once reached, will never be abandoned by the system (unless an exogenous shock displaces it).

Then the relationship between actual prices and production prices is almost always[1] studied in terms of *asymptotic stability* of the equilibrium.

An attractive alternative approach is to study the (asymptotic) stability of a *closed orbit* surrounding the equilibrium point. This would incorporate the idea of actual prices moving ("gravitating") around production prices.

The introduction of stochastic elements into the system (and in to the notion of equilibrium itself) could also improve the realism of the model. Unfortunately both these steps have a cost in terms of possibility of getting definite results.[2]

The last issue I want to address in these introductory remarks concerns the *definition* of production prices. Sometimes their definition appears to be based on a formal property: the uniformity of the rate of profit across sector.

In my opinion, it should be based on an *economic* property: that the profit rates do not determine net movements of capital across sectors or away from the production sphere.

Since capital movements are slow, they can be considered — as they always have in the tradition of economics — as *long-run* phenomena. Then, the (far from new) definition I shall adopt is as follows: production prices are "long-period equilibrium prices, in the sense that they do not determine net movements of capital across sectors or away from production".[3]

## 2. CROSS-DUAL MODELS

2.1. In this section, in order to survey the cross-dual models, we shall build a fairly general analytic structure which can encompass most of them.[4]

Let us denote by $q_t$ the n-dimensional vector of output levels at time $t$. Then the production model we shall use in this section is as follows:

required inputs (including wage-goods paid to the workers and stocks) at time $t$:

$$B'q_t$$

output + residual stocks at time $(t+m)$:

$$(I + B' - A')q_t$$

[1] The only exception is FRANKE (1988).

[2] For instance, FRANKE shows the stability of closed orbits surrounding the equilibrium, but the assumptions required (see Section 4 *below*) are very strong.

[3] Obviously, the former definition is equivalent to the latter when free entry prevails everywhere. But the latter also covers situations in which barriers to entry exist and net capital movements take place only when differentials exceed certain levels.

[4] A more detailed presentation of this structure and full proofs of all the results mentioned in this section can be found in Boggio (forthcoming).

$B$ and $A$ are square matrices of order $n$, $I$ is the identity matrix in $R^n$; moreover,

$$B \geqq A \geq 0$$

$A$ is indecomposable and productive, $m > 0$ is the "period of production", *i.e.* the interval between inputs and outputs. By setting $B = A$ and $m = 1$ we get the "circulating capital" model of production. If $B \geq A$, we also have durable inputs, say "stocks".

A typical cross-dual model, then, is a two-equation system (each equation in $R^n$) of the following kind.

*Dynamics of output*

(I) $$q_{t+m} - q_t = q_t^d f(r_t, r_t - r_{at}e)$$

The suffix $^d$ denotes a diagonalized vector, $r_t$ and $r_{at}$ are the vector of profit rates and the average rate of profit, respectively; e: = (1,1, ... 1). The former is a function of prices, the latter of both prices and output levels. $f$ is a continuously differentiable function from $R^{2n}$ to $R^n$ with $\frac{\partial f_i}{\partial r_j}$ and $\frac{\partial f_i}{\partial (r_{jt} - r_{at})}$ are both positive if $i = j$, null if $i \neq j$.

The economic interpretation of equation (I) is then that each rate of growth is an increasing function of both the rate of profit of the sector and the differences between sectoral and average rate of profit.

*Dynamics of prices*

(II) $$p_{t+h} - p_t = h\, p_t^d\, u(v_t)$$

$p_t$ and $v_t$ denote the vector of, respectively, prices and the ratios between excess-demand and output. More precisely, the latter is defined by

(III) $$v_t := (q_t^d)^{-1} (B'q_{t+m} - B'q_t + A'q_t + c_t - q_t)$$

$c_t$ is the vector of consumption out of profits (workers consume the wage-goods paid to them and included in $B$); $u = (u_i)$ and $u_i$ is a sign-preserving continuously differentiable function of $v_{it}$; $h$ is a positive parameter, setting the frequency $(1/h)$ of price changes. Thus the economic meaning of equation (II) is that the proportional rate of price changes has the same sign of excess-demand.

If we also assume that

$$c_t = c(p_t, q_t)$$

where c is a continuously differentiable function from $R^{2n}$ to $R^n$, we can see that (I) and (II) form a dynamical system where the vector of prices and output levels at time t depend only on these vectors at previous dates.

The exact nature of the dynamical system depends on the way we specify m and h. The four basic possibilities are as follows.

1) $h = m = 1$

2) $h = m =$ infinitesimal.

Then equations (I) and (II) form, respectively, a system of difference and differential equations in $R^{2n}$.

These are the two kinds of model commonly found in the literature. They can both be criticized for imposing equality between the period of production and the interval between price changes. Price changes should take place at very short intervals or in continuous time, whilst the period of production should be a larger interval, certainly non-infinitesimal. Therefore one should adopt one of the two following specifications:

3) $h$ infinitesimal, $m$ non-infinitesimal

4) $h$ non-infinitesimal, $m/h$ a positive integer, say $\Omega$.

Then we get, respectively, a mixed difference — differential equation system and a difference system in $R^{(\Omega+1)n}$. We shall call these four specifications: simplified-discrete (SD), simplified-continuous (SC), full-continuous (FC) and full-discrete (FD) time models, respectively.

Under usual (and suitable) assumptions,[5] in each of these four models two vectors $q^* > 0$ and $p^* > 0$ and a scalar $g > 1$ can be defined, such that if $q_0 = \beta\, q^*$ and $p_o = \alpha\, p^*$, then $q_t = \beta\, G(t) q^*$ and $p_t = \alpha\, p^*$, all $t > 0$. Here $\alpha$ and $\beta$ are arbitrary positive scalars and $G(t)$ is a growth factor, being equal to $e^{(g-1)t}$ in the SC-model, to $g^{t/m}$ in the other three models.

The four models can then be re-written using the transformation

(T) $$x_t := (G(t))^{-1} q_t$$

$x_t$ is the vector of output levels "discounted" by the growth factor.

For the *transformed* systems, the set

$$\{\alpha\, p^*, \beta\, q^*;\ \alpha, \beta \in R_{++}\}$$

is an *equilibrium set.* $p^*$ is a vector of production prices, $q^*$ is a vector of balanced growth proportions and $(g - 1)$ is the equilibrium rate of growth of output levels.

Models of the SD- and SC-type can be found in Boggio (1985) and Duménil et Lévy (1987) and in Franke (1986) (1987) and Kuroki (1986), respectively.

2.2. In the literature we also find other versions of the above models. They can be classified in two categories:

i) "temporary" or "short period equilibrium" (SPE) versions

ii) "rationing procedure" (RP) versions.

[5] For instance, Say's law and a consumption function homogeneous of zero degree in $p_t$ and of first degree in $q_t$.

The former assume that at each date demand equals supply in each market so that eq. (II) is replaced by

(IV) $$v_t = 0$$

It can be shown that this approach is an acceptable approximation to system (I)-(II), provided that

1) the speed of adjustment of prices is very high relative to those of prices
2) for fixed output, the short period equilibrium — implicitly defined by (III) and (IV) — is asymptotically stable.

SPE versions of either the SC- or the SD-model can be found in Nikaido (1985), Kuroki (1986), Dutt (1988), Franke (1987).

The rationing procedure versions assume that, for each good, either *ex-ante* investment (Nikaido, 1983 and 1985) or *ex-ante* consumption (Kubin, 1989) are *rationed*, so that their *ex-post* amounts are brought to equality with the difference between output and either consumption or investment, respectively. The rationing is effected by price changes that do not affect the *ex-ante* monetary amount of the rationed magnitude, but simply its equivalent in terms of goods.[6] Let us denote, respectively, by $i_{it}$ and $c_{it}$ *ex-ante* investment and consumption for the *i-th* good at time *t*. Then the two rationing procedures can be formalized as follows:

Investment rationing:

(Va) $$q_{it} - c_{it} = p_{it}\, i_{it} / p_{it+1}$$

Consumption rationing:

(Vb) $$q_{it} - i_{it} = p_{it}\, c_{it} / p_{it+1}$$

$p_{it}\, i_{it}$ and $p_{it}\, c_{it}$ are the *ex ante* magnitudes in money terms.

We shall not dwell here on the arguments used by te two authors to support their approaches. The assumption that just one of the two components of demand is rationed is certainly somewhat arbitrary. One may also wonder who is actually effecting the rationing. It must be a stronger version of the Walrasian auctioneer, taking care not only of price changes but also of quantity assignments.

The merit of this approach, however, is that it faces, in particular, the problems arising when the planned output vector is not feasible. The only acceptable alternative to a rationing procedure is to introduce inventories and model their behaviour. This would require an additional dynamic equation for each input good.

Finally, notice that eq. (V) implies $(p_{it+1} - p_{it}) / p_{it+1} = (c_{it} + i_{it} + q_{it}) / (i_{it} \text{ or } c_{it})$ i. e. that price changes have the same sign of excess-demand.

[6] This implies that, the ex-ante rationed magnitude in money terms does not react immediately to price changes.

2.3. We shall now summarize certain results that can be reached within the above analytical structure. Although some were already obtained in the literature we have quoted above, full proofs for all of them are given in Boggio (forthcoming).

2.3.1. *Results on cross dual models without consumption out of profit.* When discrete time models are used we always find instability results: strong instability for the SD-model, if $A = B$ and $n = 2$; simple instability for the FD-model, if $A = B$.

On the contrary, when the simplified continuous time model is used, we also find cases of asymptotic stability: when $A = B$ and $n = 2$ then the equilibrium is asymptotically stable if det $A < 0$, strongly unstable if det $A > 0$. This different behaviour of discrete *versus* continuous time model is puzzling.

In order to shed light on this issue, we shall compare the two simplified models with the corresponding full model. In general this approach is made very difficult by the formidable task of studying the FC-model in its basic version, which is a mixed difference — differential system. In two cases, however, this difficulty disappears: in both the SPE and RP versions, if there is no consumption out of profits, the FC-model is reduced to the difference system

(VIa) $$B'q_{t+m} - B'q_t + A'q_t - q_t = 0$$

(Notice that, given a function of $t$ defined in the interval $[0, m]$ of the real line, this equation traces out the trajectory of $q_t$ for every $t \geqq m$).

Obviously, equation (VIa) also holds for the FD-model and, with $m = 1$, for the SD-model. On the contrary, the corresponding equation for the SC-model is

(VIb) $$B' \frac{dq}{dt} = (I - A)\, q_t$$

These two equations exhibit rather different behaviour. If $A = B$, $n = 2$ and det $A \neq 0$, the "discounted" output vector $x_t$ under eq. (VIa) — transformed by (T) — always diverges from the equilibrium set $\{\beta q^*; \beta \in R_{++}\}$; under eq. (VIa) — transformed by (T) — $x_t$ converges to the equlibrium set if det $A < 0$, diverges if det $A > 0$.

We can then, draw the fundamental conclusion that since the simplified continuous time model behaves very differently from the full continuous time model, it must be rejected. Clearly, the reduction of the period of production to an infinitesimal quantity gives rise to a fatal distortion in the dynamic behaviour of our variables.

Having rejected the SC-model on these grounds, we are left with instability results only.

2.2.2. *Results on cross dual models with consumption out of profit.* Let the value of consumption out of profit be a fixed proportion of profits accruing to the firms if all output is sold (when they are non-negative). Two main alternative approaches are then open:

*a*) To assume price substitution effects. In this case, further assumptions — like gross substitutability — can ensure that the Jacobian matrix at an equilibrium point is negative semidefinite. This, as we shall see, is a crucial property (that we shall denote by NSJ).

*b*) To assume that consumption out of profits is a scalar multiple of fixed vector: the "fixed-proportion consumption function". A special case of this alternative is the widely used two-sector model with one consumption good and one investment good. When capitalists consume only one consumption good, as in this special case, NSJ is guaranteed. But in the general case it may prevail or not, depending on the proportions between different goods in the consumption basket.

The important consequences of these facts are fully brought out by the following results, which refer to the SD-model.

1) When n = 2 or the saving propensity and the effect of profit rates differential on growth rate are both sufficiently small, NSJ is a necessary condition for stability.

2) When the above conditions are both fulfilled and the price reaction coefficients are not "too large", NSJ also becomes a sufficient condition for stability.

Therefore we may conclude that when consumption out of profits is introduced into the discrete time models, their behaviour is more likely to be stable if the NSJ condition is fulfilled. But this condition is not satisfied in general by the fixed proportion consumption function. Moreover, the assumption that capitalists consume one good only (a sufficient condition for NSJ) may look plausible in the context of two sector models, but not when n is much larger than 2.

## 3. FULL-COST MODELS

3.1. In the full-cost models, the price is determined by the cost of production — measured at the normal level of utilization of capacity — plus a target rate of return.

For instance, in the context of Leontief technology with production period equal to 1 and without fixed capital, a typical dynamic equation of price determination under full-cost is the following:

(I) $$p_{it+1} = (1 + \pi_i)\,(A_i\, p_t + w_t\, l_i) \qquad i = 1, 2, \ldots n,$$

where $A_i$, $l_i$, $p_t$, $\pi_i$, $w_t$ are the vector of input coefficient of goods and the labour input coefficients in the *i-th* sector, the target rate of return in that sector and the wage rate at time t, respectively.

The right hand side of (I) can be called "full-cost function" (of the *i-th* sector). A general formulation of the full-cost assumption is then

(II) $$p_{it+1} - p_{it} = \alpha_i \, (f_{it} - p_{it}) \qquad 1 \geqq \alpha_i > 0$$

where $f_{it}$ is a generic "full-cost function". It can easily be shown that if $w_t$ is constant — say, equal to $w$ — and if the target rates of return are uniform across sectors — say, equal to $r$ — and satisfy

(III) $$0 < (1 + r) < (R\,(A))^{-1}$$

where $R\,(A)$ is the spectral ray of $A$, then, under eq. (I), the price vector converges to the vector of production prices

$$(I - (1 + r)\; a)^{-1}\; (1 + r)\; w\; l$$

This is a straightforward consequence of the properties of non-negative matrices. These properties also lead to convergence results with many other different and more complicated formulations of the full-cost function.[7] Further extensions are then allowed by the application of certain non-linear generalizations of non-negative linear operators.[8]

However, under certain conditions, the convergence of prices may exhibit different features. To illustrate the issue, let us reconsider eq. (I) and assume $A$ primitive and

$$(1 + r) > (r(A))^{-1}.$$

Then all absolute prices will diverge, but the relative prices may well converge: the real wage will shrink towards zero, and the behaviour of the system will asymptotically approach that of the homogeneous equation

(VI) $$p_{t+1} = (1 + r)\; A\; p_t$$

As is well-known, in this case the normalized price (*i.e.* relative price) vector, $p_t / \|p_t\|$ will always converge to the positive column eigenvector of $A$ having norm equal to 1. Notice that an equation like (VI) arises directly when the wage rate is always equal to the current value of a given basket of goods.

Things get even more complicated, if, for instance, we assume the following full-cost function:

(VII) $$f_t = Q(r)\; p_t$$
$$Q := ((1 + r)\; A + rB)$$

[7] For a detailed presentation of models and results on full-cost models, cf. Boggio and Gozzi (forthcoming), pp. 5-6.

[8] For instance, those expounded in Nikaido (1968), ch. III.

which is appropriate in the context of the production model assumed in Section 2 of this paper.

Then, if $R(Q(r)) > 1$ and $Q(r)$ is primitive, we still have asymptotic convergence of $p_t/||p_t||$ to the normalized positive column eigenvector of $Q(r)$, say $p^o$. It is easy to show that, in general, $p^o$ changes with $r$. We shall then use the notation $p^o(r)$.

Let us define now the *ex-post* real rates of profit as "the rate of growth of the value of the invested capital in real terms, when all profits are immediately reinvested".

In the case of eq. (VII) the *ex-post* real of profit are a vector $\sigma \in R^n$, solving

$$p^o(r) = ((I + \sigma^d)\ A + \sigma^d\ B)\ p^o(r)$$

If the definition of production prices as long-period equilibrium prices — in the sense specified in Section I — is accepted and we suppose the case when the vector of *ex-post* real rates of profits that do not determine net movements of capital across sectors is the uniform rate vector, then this vector will prevail only for those particular values of $x \in R$ that solve the equality

$$p^o(r) = ((1 + x)\ A + x\ B)\ p^o(r) \qquad r,x \geqq 0$$

An obvious solution is $x = r = r^O$ and $r^O$ is a value of $r$ (which exists) that makes $R(Q(r)) = 1$.

This might not be the only solution, but certainly the solution set is not the entire interval $[\ 0, \infty)$.

Thus, if firms do not "choose" a value of the target rate of return $r$ belonging to that solution set, *relative prices will indeed converge to a well-defined vector, but this will not be the vector of production prices.*

Some sectors might even earn a negative *ex-post* real rate and will not be able to reproduce themselves. If they produce basic commodities, the system itself will gradually disappear.

This absurd possibility could be avoided if, in the formation of target rates of return, a role were played by (long-run) supply and demand conditions. This last remark leads us to consider the more general shortcoming of these models as they stand: they embody a short-period theory of price formation, not supplemented by a (long-period) theory of target rates of return formation.[9]

Therefore by their very nature, they cannot describe the relationship between long-period equilibrium prices — as are production prices — and actual prices.

[9] With one exception (Boggio, 1986) no attempt has been made in that direction.

## 4. CONCLUDING REMARKS

We can summarize our main conclusions about the class of cross-dual models we have examined in Section 2 in the following way.

*a)* The simplified continuous time models should be abandoned because the reduction of the production period to an infinitesimal distorts their asymptotic behaviour.
(This obviously also holds for their short-period equilibrium and rationing procedure versions).

*b)* The discrete time models without consumption out of profits are unstable.

*c)* When consumption out of profits is introduced into the latter, results more favorable to stability can be obtained, if the Jacobian matrix of the consumption function at the equilibrium points is negative semidefinite (NSJ). In particular this condition is necessary for stability in meaningful cases.

*d)* If we neglect the totally implausible case when only one good is consumed by capitalists, the NSJ property is not guaranteed by the constant proportion consumption function. To obtain NSJ, one should then rely on price substitution effects and some additional assumption like gross-substitutability, whose Neoclassical flavour is certainly unpleasant for most students of the theory of production prices. It is also natural to notice that, by moving in this direction, the stability conditions for these models become very similar to those of orthodox general equilibrium theory.

We shall now devote some brief comments to other cross-dual models, not (completely) amenable to the analytic structure on which we have based the results of Section 2.

We mention here the papers by Flaschel and Semmler (1986) (1987), Franke (1988), Kubin (1989), Duménil et Lévy (1989). In the papers by Flaschel and Semmler and by Franke the basic cross-dual model is of the pure differential equation type, therefore subject to the fundamental critique to this kind of model raised in Section 2.

Moreover, they adopt the very strong assumption that the planned rates of growth of output appearing in the equation determining price dynamics, and the average rate of profit appearing in the equation determining output dynamics, are the equilibrium rates.

For this model the (non-asymptotic) stability of the equilibrium set (by Flaschel and Semmler) and, more generally, of any member of a family of closed orbits including the equilibrium ones (by Franke), is proved.

Flaschel and Semmler, by adding the assumption that output growth depends on the rate of change of profits, can also prove the asymptotic stability of the equilibrium set.

Kubin's model is a rationing procedure version (cf. eq. (Vb) of Section

2) of a simplified discrete time model with fixed proportion consumption out of profits, and with the additional assumption that the planned rates of growth of output also depend on relative output levels. Asymptotic stability is proved for certain intervals of the relevant parameters.

A remarkable step forward in the direction of realism and plausibility is made in the paper by Duménil et Lévy, in particular because they also model the behaviour of inventories and the degree of utilization of capacity.

Unfortunately, they can prove asymptotic stability only for very small reaction coefficients of prices and outputs to excess-demand.

Turning now to the full-cost models, I should like to recall first of all their points of strength:

*a*) the very robust convergence results;

*b*) the empirical evidence on price formation in manufacturing, which gives a most important role to cost changes and only a minor one to demand conditions.

As to their shortcomings, the main one is the (almost) complete absence of explanations of the formation of target rates of return; secondly, that no role is given to supply-demand conditions.

I believe that, if we want to build a dynamic out-of-equilibrium theory of long-period prices of produced goods (in particular manufactured goods), the most promising way forward, in terms both of realism and of definiteness of results, is to develop the full-cost models so as to overcome those shortcomings.

4.2. The final point I want to address is a purely methodological one, that some people perhaps would have wished to see as the opening question of a paper like this. By means of these or similar models, can we really prove or disprove the factual relevance of production prices?

This is just a special case of a more general problem. Given a certain equilibrium theory, can we prove or disprove it by means of out-of-equilibrium dynamic theories?

The answer, strictly speaking, must be negative. A particular vector can be an equilibrium for wide classes of dynamic models and one can never be sure that all the relevant possibilities have already been discovered and fully explored. The more one is aware of the relative impotence of matematical tools, in most cases, to give definite results and of the inability of practitioners to use those tools to their full potential, the more cogent becomes the above observation.

On the other hand, we must certainly recognize that stability (instability) results, obtained within plausible dynamic models, must influence the degree of subjective credibility of the relevant equilibrium theory.

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