political economy *Studies in the Surplus Approach*

volume 6, numbers 1-2, 1990

Production Prices and Dynamic Stability: Comment on Boggio

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I think that the work Boggio has presented should be taken as an important basis for future discussion on gravitation. He has given both quite a complete taxonomy of models that have been set out and explored, and many results on the behaviour of such models near their long-run position. As I have no reason for major criticism, my comment will be just a complementary short paper, aiming at an elementary illustration of the following points: (1) The formal models analysed by Boggio and many other scholars in this area represent a reasonable formalisation of the ideas shared by classical economists about the forces that should ensure convergence of market prices toward natural prices. (2) However, once translated into formal terms a naïve element of those ideas and reasonings becomes apparent: in fact classical economists were not able to fully understand the working of a dynamic feedback system; such a weakness, as I will try to show, can explain the reason why convergence was considered as an almost obvious feature of market prices by classical economists, whereas modern analysis has mainly obtained negative results. (3) I will also argue that lack of full understanding of feedbacks seems to be a common shortcoming of both classical dynamic model of market prices and Walrasian tâtonnement, and that both cases rapresent failures of the idea of an invisible hand taking market economy toward its long-run position.

1. I shall begin by quoting a passage from Ricardo's *Principles*: "I have already remarked, that the market price of a commodity may exceed its natural or necessary price, as it may be produced in less abundance than the new demand for it requires. This however, is but a temporary effect. The high profit on capital employed in producing that commodity will naturally attract capital to that trade; and as soon as the requisite funds are supplied, and the quantity of the commodity is duly increased its price will fall, and the profits of the trade will conform to the general level".[1]

[1] David Ricardo, *On the Principles of Political Economy and Taxation*, edited by Piero Sraffa, 1951, Cambridge U.K., p. 119.

Although not stated in formal terms, this is the description of the working of a dynamic system. I will begin by rephrasing it in different steps. Let us assume, for simplicity, that the economy consists of just two industries.

a) It is assumed that at the beginning of the process production of commodity 1, for instance, is lower than "normal", or "long-run" demand.

b) It is also assumed that at the beginning of the process market price of 1 is higher than its natural price. In our two-industry economy this is equivalent to saying that the rate of profit in industry 1 is higher then in 2. Notice that here I am only interested in reproducing Ricardo's "disequilibrium" starting position, not in the reasons why the system starts in that position (these have been considered by Ricardo in Chapter IV, to which he refers at the beginning of the passage: "I have already remarked...".)

c) As an effect of *b*) capital moves toward industry 1 and therefore supply of 1 rises relative to supply of 2. This is quite a clear dynamic "equation", relating changes of supply in industries to rate of profit differentials.

d) The last part of the passage is not completely clear. Nonetheless I think it may be interpreted as saying at least that, as an effect of *c*), sooner or later the composition of supply relative to composition of demand will cause a fall of price of 1 relative to price of 2.

2. Let us now try to translate the above steps, in particular *c*) and *d*), into the simplest formal model. Let:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be the matrix of the industries' requirements per unit of output (requirements for industry i are the coefficients on the i-th row). Constant returns to scale are assumed. Moreover: (i) commodity 1 is a means of production, employed in positive quantities in both industries; (ii) commodity 2 is the consumption good; (iii) coefficients a_{12} and a_{22} are both positive and represent real wage per unit of output (labour per unit of output, times consumption good per unit of labour).

As A is indecomposable, Perron-Frobenius theorem ensures that it has two real eigenvalues, a and a_1, with $a > 0$, $a > |a_1|$. Moreover, a possesses a real positive eigenvector, whereas the same does not hold for a_1. Feasibility implies $a < 1$.

Prices of production are:

$$p^* = A(1 + r)p^*, \tag{1}$$

with $r = (1 - a)/a$, p^* being the positive eigenvector of A associated to a.

To further simplify the model I will assume that its long run position on the quantity side is determined as the steady state:

$$q_{t+1} = (1 + r) q_t$$

$$q_t = q_{t+1} A,$$

i.e.:

$$q_t = A(1 + r) q_t, \tag{2}$$

so that q_t is a positive row eigenvector of A associated to a. In the above equation, and below, the precise meaning of $q_{i\tau}$ is that of the quantity which industry i produces during the time between τ and $\tau + 1$, and is available at $\tau + 1$.

Notice that the steady growth assumption implies that capitalists do not consume and that the labour force grows at rate r.

Let us now come to the market price dynamics. A considerable reduction in the number of equations and variables will result from focussing on the dynamics of the ratios:

$$P_t = \frac{p_{1t}}{p_{2t}}, \quad Q_t = \frac{p_{1t}}{q_{2t}}.$$

The following equation:

$$Q_t - Q_{t-1} = \alpha[r_{1t-1} - r_{2t-1}]$$

$$= \alpha\left[\frac{p_{t-1}}{a_{11}p_{t-1} + a_{12}} - \frac{1}{a_{21}p_{t-1} + a_{22}}\right] \tag{3}$$

where α is a positive speed of reaction coefficient, formalises the idea of an increase (decrease) in production of 1 relative to 2, if there is a positive (negative) difference between the rates of profit calculated using the market prices one period before (this is step c) of Section 1).

To complete the system we must write an equation describing the dynamics of prices:

$$P_t - P_{t-1} = -\beta\left[\frac{q_{1t-1}}{q_{2t-1}} - \frac{a_{11}q_{1t} + a_{21}q_{2t}}{a_{12}q_{1t} + a_{22}q_{2t}}\right]$$

$$= -\beta\left[Q_{t-1} - \frac{a_{11}Q_t + a_{21}}{a_{12}Q_t + a_{22}}\right], \tag{4}$$

where β is a positive speed of reaction coefficient (this is step *d*) of Section 1).

Two observations are necessary. First, consistently with the assumption on capitalist consumption, demand for good 2 is limited to the productive necessity (labour employed, times real wage). Secondly, model (3)-(4) cannot be thought of as representing an actual economic dynamics. In fact, the quantity ratio Q_t, determined in (3), and used in (4), is no more than the ration between *intended* quantities. Such a ratio can be actually obtained only if:

$$Q_{t-1} = \frac{a_{11} Q_t + a_{21}}{a_{12} Q_t + a_{22}}. \tag{5}$$

This implies that $P_t = P_{t-1}$. Thus, if equation (5) kept holding through time, we should conclude either that the system has already reached equilibrium, or that P_t has got stuck in a disequilibrium position. In fact, it may easily be argued that this second phenomenon is not possible since Q_t would change and therefore P_t would change as well. Thus in general (5) does not hold out of equilibrium, so that the alternatives are: (i) we must add equations to take into account inventories or rationing; (ii) we limit ourselves to model (3)-(4), with the proviso that we are not going beyond a sort of tâtonnement process, i.e. a model in which production and exchange may take place only when equilibrium has been established. Important work exploring the first alternative is mentioned in Boggio's paper. For the limited purpose of this comment, i.e. an illustration of the fact that classical ideas about gravitation do not lead straightforwardly to convergence, the second alternative appears to be sufficient.

3. Analysis of the Jacobian of model (3)-(4) is carried out in Appendix 2. Here let us follow an intuitive-approximative argument.

(A) Perron-Frobenius theorem implies that $r_{1t} = r_{2t}$ if and only if $P_t = P^*$ (as there is only one eigenvector associated with a, up to a proportionality coefficient). In Appendix 1 it is shown that $r_{1t} - r_{2t}$ is positive or negative according to whether P_t is greater or smaller than P^*.

(B) In the same way, putting:

$$M_t = Q_t - \frac{a_{11} Q_t + a_{21}}{a_{12} Q_t + a_{22}},$$

M_t vanishes if and only if $Q_t = Q^*$. In Appendix 1 it is shown that M_t is positive or negative acording to whether Q_t is greater or smaller than Q^*.

(C) Now assume that P_t and Q_t lie very near P^* and Q^* respectively, and that α is very small, so that the sign of

$$Q_{t-1} - \frac{a_{11} Q_t + a_{21}}{a_{12} Q_t + a_{22}},$$

which determines the dynamics of P_t, is normally the same as that of M_t (as $Q_t - Q_{t-1}$ is very small).

In Figure 1 a sketchy representation of the dynamic corresponding to facts (A), (B), (C) is given. In Figures 2 and 3 the two possible trajectories which are compatible with Figure 1 in a small neighborhood of $(Q^* \; P^*)$ are drawn. Now, before discussing the stability issue, i.e. whether the process follows an exploding or a converging spiral, it is worthwhile returning to

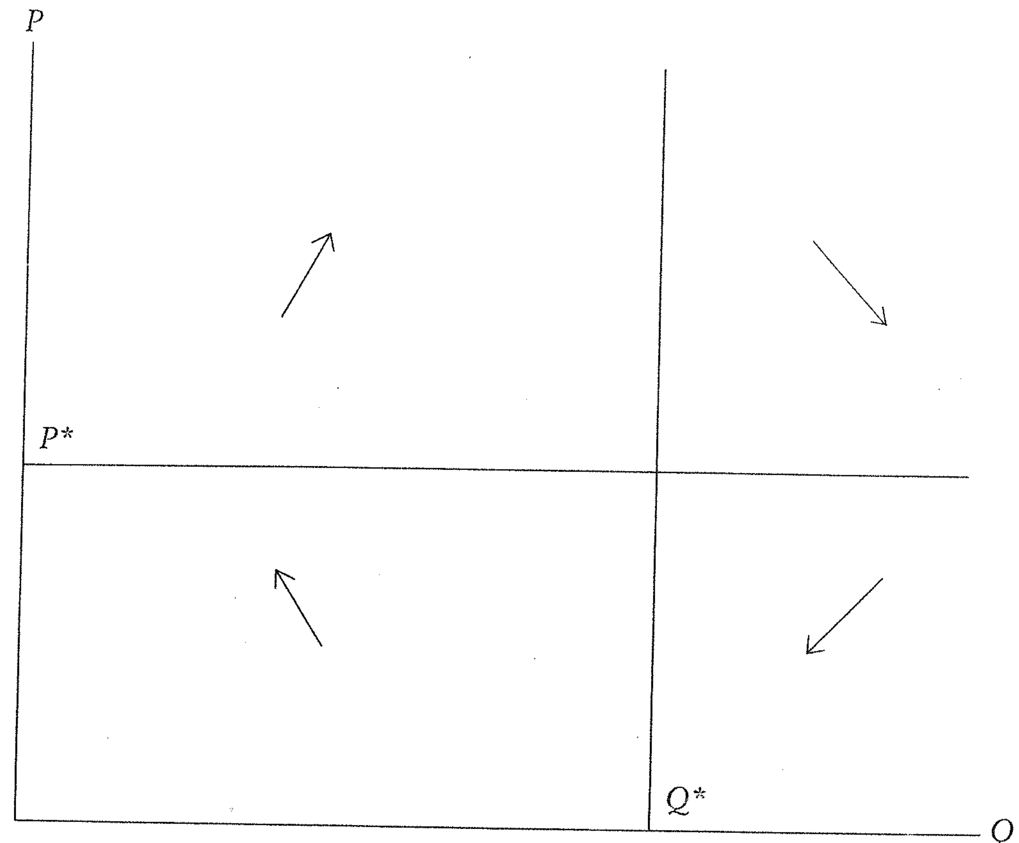

Figure 1: Dynamics of (3)-(4) near equilibrium for a small α.

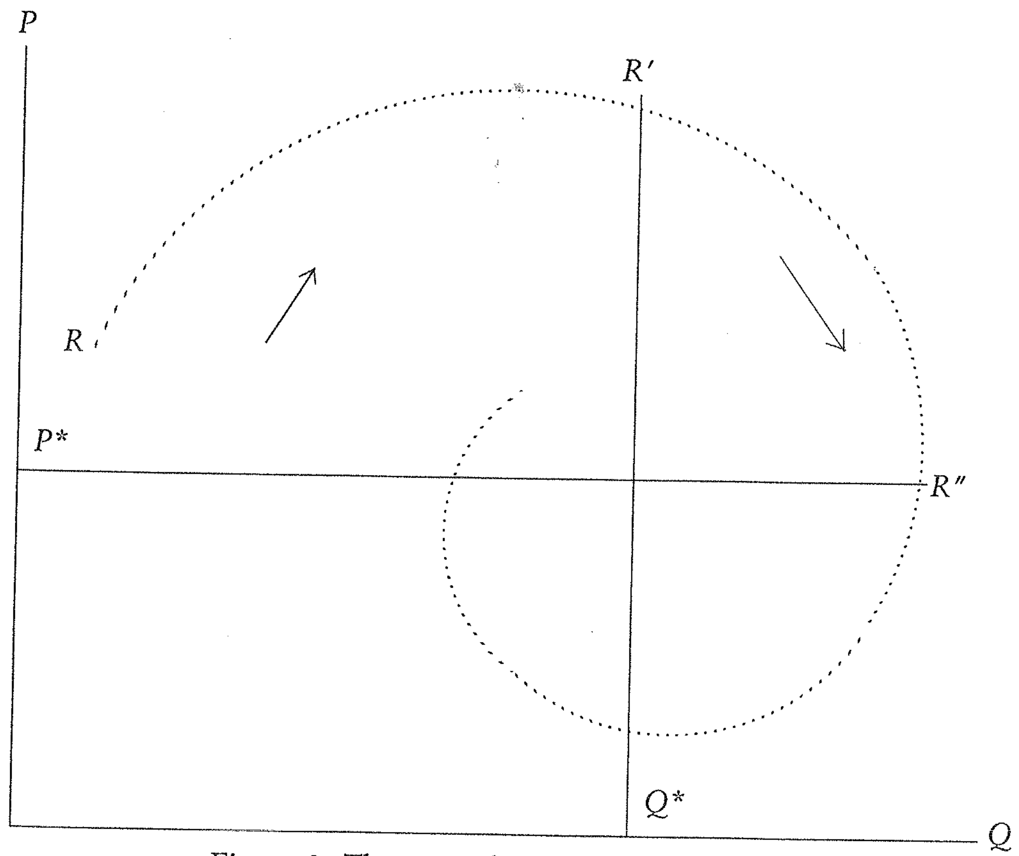

Figure 2: The case of a convergent spiral.

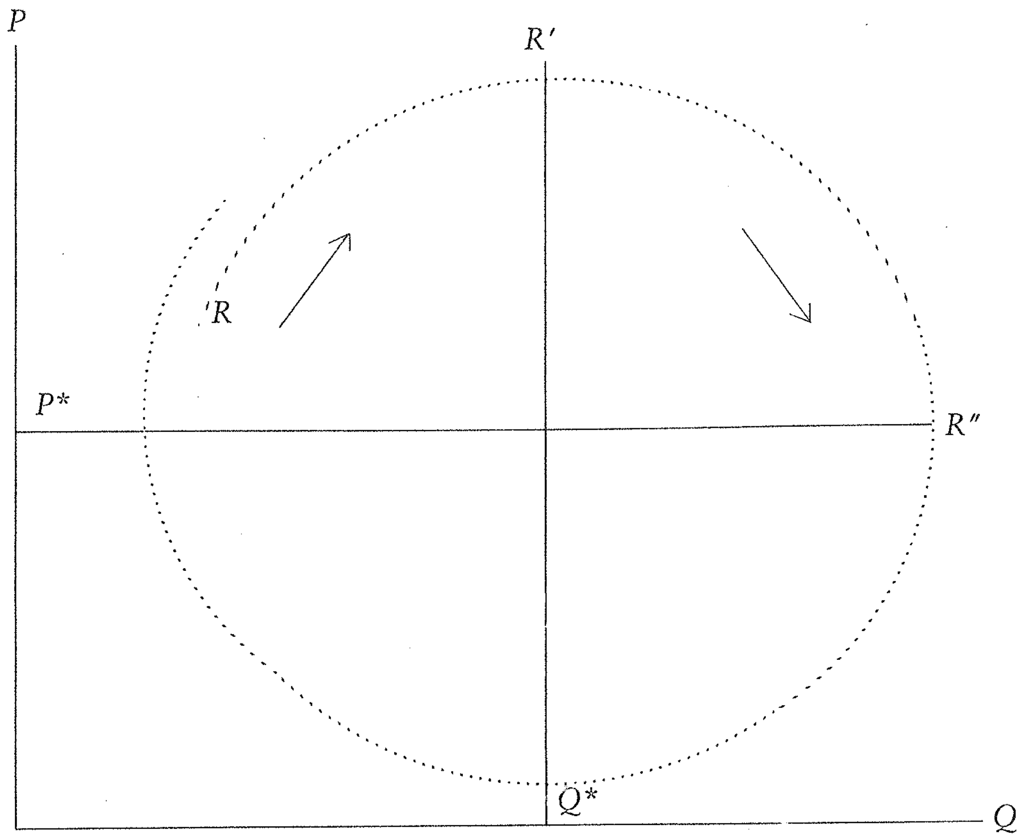

Figure 3: The case of a divergent spiral.

Ricardo's passage. In Figures 2 and 3 the starting point of Ricardo has been indicated by R (remember steps (a) and (b) in Section 1). In both cases, starting from R, quantities move in favour of the industry wherein the rate of profit is the highest. Shortly after R' has been reached excess demand has disappeared so that P_t could rest. But since ratio P_t is still higher than P^*, quantity ratio Q_t keeps rising, so that P_t begins to fall and eventually reaches P^*. However, when this occurs, Q_t has gone far beyond Q^*.

Ricardo's statement does not appear to contain any immediately apparent mistake. Nonetheless, it must be pointed out that: (i) while Q_t rises between some point to the right of R and R' the simultaneous rise of P_t is not mentioned; (ii) while P_t falls, between some point to the right of R' and R'' the simultaneous rise of Q_t is not mentioned. Thus, one has the impression that Ricardo's firm belief in convergence relies on his inability to keep under control the feedback determination of quantities and prices, or perhaps on taking the two directions of causation one at a time. Firstly high price causes quantity ratio to rise, but price ratio rises as well. Subsequently, price falls and equilibrium is reached; by contrast, quantity ratio rises *at the same time*, and equilibrium, if reached, can only occur as the limit an oscillatory process.

4. Figures 1 to 3, and the above argument, should be sufficient to clarify what has been anticipated at the beginning of this comment, namely that classical economists' ideas about gravitation suffer of a fundamental weakness due to inability to deal properly with feedback effects. Once this point has been made, so that we understand that the working of the system is more complicated than in the naïve classical representation, the result of local stability analysis of system (3)-(4) will no longer be upsetting. I mean that we should no longer be committed to the idea that market prices must converge. In fact, system (3)-(4) is divergent, as in Figure 3, independently of matrix A, no matter how small coefficients α and β are taken.

Whereas an intuitive explanation leading to a pair of complex conjugate roots has been given in Section 3, I am not able to do the same with divergence, i.e. the fact that such roots have modulus greater than 1. Hence, the interested reader can choose among the elementary calculations in Appendix 2 of this comment, Boggio's paper, or other works on the subject. Here it must be pointed out that neither the spiraling trajectories of Figures 2 and 3, nor the instability result are exclusively due to the discrete time representation chosen. This issue cannot be dealt in detail within the limits of this comment. I limit myself to insisting on the fact that the instability result turns out irrespective of the size of the speed of reaction coefficients (see Appendix 2 for some more detail), and to urging the reader interested in this aspect of the problem to consult Boggio's paper about results on continuous time models.

5. Lack of ability to deal with feedback effects appears to be a major drawback not only of classical theory of gravitation. On the other side, Walras' idea of the convergence of his tâtonnement can be shown to be at least in danger because of an important feedback which he failed to perceive. In the most simple Walrasian general equilibrium model, i.e. a pure exchange economy, when the auctioneer moves prices according to excess demand, he moves *monetary* incomes as well. This makes the signs of excess demand changes unpredictable.[2] Thus the simplest representation of the way market forces work, both within classical and neoclassical paradigms—both based on agents which blindly maximise a function of current market signals—do not lead to unconditionally converging systems. Whether this is due to the exceding poverty of those representations as such, or to the necessity for a more radical change in the conception of market forces themselves, is completely outside the limits of this comment.

[2] For a recent assessment of the literature on excess demand see A. KIRMAN, "The Intrinsic Limits of Modern Economic Theory: The Emperor Has No Clothes", *The Economic Journal*, vol. 99 (Conference Volume), pp. 126-139.

Appendix 1.

It has already been argued that $D_t = r_{1t} - r_{2t}$ has only one zero at P^*. The derivative of D_t with respect to P_t is:

$$\frac{d}{dP_t}\left[\frac{P_t}{a_{11}P_t + a_{12}} - \frac{1}{a_{21}P_t + a_{22}}\right] = \frac{a_{12}}{(a_{11}P_t + a_{12})^2} + \frac{a_{21}}{(a_{21}P_t + a_{22})^2}$$

and is therefore positive for any P_t. Hence D_t is a monotonically increasing function of P_t. The above derivative at P^* will be indicated by γ.

Let us now consider the derivative of M_t with respect to Q_t:

$$\frac{d}{dQ_t}\left[Q_t - \frac{a_{11}Q_t + a_{21}}{a_{12}Q_t + a_{22}}\right] = 1 - \frac{\det(A)}{(a_{12}Q_t + a_{22})^2}.$$

As

$$A = B^{-1}\begin{pmatrix} a & 0 \\ 0 & a_1 \end{pmatrix} B,$$

for a suitable non-singular matrix B, $\det(A) = a\, a_1$. Moreover, as

$$A'(1+r)\begin{pmatrix} Q^* \\ 1 \end{pmatrix} = \begin{pmatrix} Q^* \\ 1 \end{pmatrix},$$

whose second row is

$$(a_{12}Q^* + a_{22})(1 + r) = 1,$$

the above derivative at Q^* is

$$1 - \frac{a\, a_1}{(a_{12}Q^* + a_{22})^2} = 1 - a\, a_1(1 + r)^2 = 1 - \frac{a_1}{a}.$$

Indicating it by δ, since $|a_1| < a$,

$$\delta = 1 - \frac{a_1}{a} > 0$$

(notice that δ is positive irrepective of whether det (A) is positive or negative); hence, as M_t has only one zero at Q^*, $M_t > 0$ for $Q_t > Q^*$.

Appendix 2.

Linearization of system (3)-(4) around (Q^* P^*) gives:

$$Q_t - Q_{t-1} = \alpha\gamma P_{t-1} + \ldots$$

$$P_t - P_{t-1} = -\beta Q_{t-1} + \beta\frac{\det(A)}{(a_{12}Q^* + a_{22})^2}Q_t + \ldots$$

$$= -\beta Q_{t-1} + \beta\frac{a_1}{a}Q_t + \ldots$$

Eliminating Q_t from the second equation gives:

$$Q_t - Q_{t-1} = \alpha\gamma P_{t-1} + \ldots$$
$$P_t - P_{t-1} = -\beta\delta Q_{t-1} + \alpha\beta\gamma(1-\delta)P_{t-1} + \ldots$$

The Jacobian is $J = I + J'$, with:

$$J' = \begin{pmatrix} 0 & \alpha\gamma \\ -\beta\delta & \alpha\beta\gamma(1-\delta) \end{pmatrix}$$

The eigenvalues of J are:

$$1 + \frac{\alpha\beta\gamma(1-\delta)}{2} \pm \frac{1}{2}\sqrt{[\alpha\beta\gamma(1-\delta)]^2 - 4\alpha\beta\gamma\delta}$$
$$= 1 + \frac{\alpha\beta\gamma(1-\delta)}{2} \pm \frac{1}{2}\sqrt{\alpha\beta\gamma\,[\alpha\beta\gamma(1-\delta)^2 - 4\delta]}\,.$$

As $\delta > 0$, taking $\alpha\beta$ sufficiently small gives a negative $\alpha\beta\gamma(1-\delta)^2 - 4\delta$, hence a negative discriminant, *i.e.* complex conjugate roots, thus confirming the spiraling pattern resulting from informal analysis of Section 3.

Coming to the modulus of the roots, if the discriminant is smaller than zero, the modulus is:

$$\left[1 + \frac{\alpha\beta\gamma(1-\delta)}{2}\right]^2 + \frac{1}{4}\left[4\alpha\beta\gamma\delta - [\alpha\beta\gamma(1-\delta)]^2\right]$$
$$= 1 + \alpha\beta\gamma > 1.$$

If the discriminant is zero the roots coincide:

$$1 + \frac{\alpha\beta\gamma(1-\delta)}{2},$$

their modulus being of course the same as in the case of negative discriminant, *i.e.* $1 + \alpha\beta\gamma$. (to obtain this result notice that vanishing of the discriminant is equivalent to $\alpha\beta\gamma(1-\delta)^2 = 4\delta$).

If the discriminant is zero or bigger than zero consider first the case $1 - \delta > 0$, *i.e.* $\det(A) > 0$. Of course the root:

$$1 + \frac{\alpha\beta\gamma(1-\delta)}{2} + \sqrt{\Delta}$$

is positive and greater than 1.

Secondly, consider the case $1 - \delta < 0$, *i.e.* $\det(A) < 0$. If the discriminant is zero, the (double) root

$$1 + \frac{\alpha\beta\gamma(1-\delta)}{2},$$

having still modulus $1 + \alpha\beta\gamma > 1$, must be negative and smaller that -1.

Raising $\alpha\beta$, so that the discriminant from zero becomes positive, the root:

$$1 + \frac{\alpha\beta\gamma(1-\delta)}{2} - \sqrt{\Delta}$$

will move even farther from -1 in the negative direction.

This completes the analysis of the eigenvalues. Instability is obtained irrespective of α, β and the sign of det(A). Incidentally, in a previous paper published in this Journal,[3] I made an attempt to give an elementary explanation of the possibility of instability in classical market price models by using a model only slightly more complicated than the present one. Unfortunately, a mistake in the calculation of the characteristic equation coefficients led me to the conclusion that instability depends on the sign of det(A). This is false for discrete time models. Even though I must recognize that a mistake is a mistake, the intuitive explanation I had put forward for the possibility of instability for market price models appears to be confirmed.

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[3] "Joint Production in SRAFFA: Some Open Issues", 1988, vol. 4, 2, pp. 213-222.