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Part III

Cross-Dual Dynamics, Derivative Control and Global Stability: A Neoclassical Presentation of a Classical Theme

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1. INTRODUCTION

In this paper we continue an investigation on the stability of Walras' cross-dual dynamics for production economies we started in Flaschel (1991). This dynamics – though formulated as a tâtonnement process – is very similar in its formal structure to the classical characterization of capital movements due to profitability differentials and the effects these movements have on market price formation. This paper can therefore also be viewed as a preliminary study of some basic dynamic features of the Classical view on the convergence of market prices towards prices of production or the Marxian view of the gravitation of the former around the latter.

Yet, since we start from the neoclassical general equilibrium framework, it must be stressed right from the beginning that this paper does not consider prices of production and the formation of normal or average profitability in a capitalist economy and that it – by its use of tâtonnement adjustment processes – also ignores many feedback mechanisms as they certainly exist in real world economies. The advantage of such an approach, however, is that the model we shall use in the following is well documented and understood – so that it will be sufficient as well as more suggestive to consider it in its simple form: the one-input one-output case.[1] Furthermore, reducing the problem of a cross-dual dynamic process to a tâtonnement procedure may help to lay bare the essentials of such a process for which, in our view, there does not yet exist a generally accepted description. This paper, for example, will investigate in particular the stabilizing contribution of so-called derivative forces, which have rarely been considered in the literature so far. In order to study their effect *in isolation*, a tâtonnement procedure may thus be a good starting point from the methodological point

* I have to thank G. Duménil, R. Franke, R. Goodwin and W. Semmler for helpful comments and suggestions. Usual caveats apply.

[1] All propositions in the present paper also hold for the multi-sectoral general equilibrium models [as they are used] in Mas-Colell (1986).

of view. Finally, global results can rarely be achieved for dynamic mechanisms which attempt to incorporate further important features of the real world — as the literature on classical gravitation processes in particular shows. It may therefore be of interest to find certain prototype mechanisms for which *global* stability can indeed be proved [as in Flaschel/Semmler (1987) or — from a quite different perspective — in the present paper] and to study more 'realistic' versions of such processes by means of computer simulations later on.

The following section will provide an introduction into cross-dual dynamics as well as its automatic derivative control by making use of the one-input one-output model of general equilibrium theory. Section 3 will then study global properties of the derivatively controlled cross-dual dynamics by means of an appropriate Liapunov function. Section 4, finally, will present some computer simulations for the example that is used in Mas-Colell (1986) by adding various derivative control mechanisms to the well-known impacts of excess demand and excess profitability. In this way a variety of stability scenarios can be created – depending on the adjustment functions that are used for derivative control. This final section therefore indicates that the choice of (variable) adjustment speed may represent a crucial step in the further analysis of the stability of capitalistic market economies.

2. WALRAS' CROSS-DUAL DYNAMICS AND DERIVATIVE CONTROL

In Mas-Colell (1986) the one-input-one-output case of general equilibrium models is used as an example to illustrate some global stability properties of a cross-dual type of Walrasian tâtonnement procedure [motivated by Mas-Colell by referencing to Walras' writings on disequilibrium in production economies]. With regard to its local properties this dynamic process is formulated and analyzed in Mas-Colell's article in great generality and detail. But with respect to global questions no general conclusions are reached in his article. For the one-input one-output case of general equilibrium analysis some interesting features are, however, noted by Mas-Colell.[2]

We shall reconsider in the following this dynamics for this simplest of all general equilibrium models, too — in order to design, explore, and illustrate quite natural and important extension of the above cross-dual dynamics in a way as instructive as possible.[3] In contrast to Mas-Colell's

[2] Note here that disequilibrium profits are neglected in Mas-Colell's partial-equilibrium investigation of this basic situation. These profits will be included in our presentation of this basic situation.

[3] The propositions which we shall formulate in this article, however, all hold for multi-sectoral economies as well, as can easily be shown by applying the following proofs to the model used in Mas-Colell (1986).

statements on the stability of Walras' cross-dual dynamics our extended adjustment process will exhibit very strong global stability properties when derivative forces are assumed to exist globally, i.e. here, at least for larger discrepancies in profitability or in supply and demand.

In Flaschel (1991) we have shown that cross-dual dynamics can lead to universal local asymptotic stability (independent of economic structure) when derivative forces are added in a particular way (by means of a single adjustment parameter). In the present paper we shall show how similar results can also be achieved for the global point of view and specific adjustment coefficients. However, such an attempt to prove global *asymptotic* stability may not be sensible from two points of view

- the choice of the size of parameters to achieve global asymptotic stability may be too extreme
- the derivative forces used to improve the stability features of the cross-dual dynamics may be operative only for larger discrepancies in the basic magnitudes that govern the laws of motion of the system thus leading to global, but not to global *asymptotic* stability.

This latter argument indeed is an important one,[4] since dynamic reactions should not be fine-tuned to such an extent that insignificant differences are assumed to give rise to distinguishable reactions of the economic agents. In particular from a classical perspective it is very plausible that profit-differentials must reach a certain critical level before their rate of change exercises a significant influence on the capital movements initiated by these discrepancies. We shall therefore also attempt to take this latter point into account in the following extension of Mas-Colell's cross-dual dynamics by means of derivative forces.

Let us start by summarizing the one-input one-output case of general equilibrium analysis and its Walrasian price/ quantity adjustment procedure (a more detailed, but still partial version of it can be found in Beckmann-Ryder (1969) and Mas-Colell (1986, pp. 64-67). We assume as given an economy where commodities are produced solely by means of labor subject to a smooth production function $f(l^d) = y^s$ which may exhibit decreasing, constant, or increasing returns to scale. Furthermore, we assume as given a smooth demand function $d(p\ \pi)$ for the one produced commodity, where profits π are defined by $\pi = p\ f(l^d) - wl^d$ (p the commodity price and w the nominal wage rate, $w = 1$ by choice of numéraire). Households' initial endowments consist of labor solely and labor supply can be derived from the above demand function by means of Walras' Law

$$pd(p,\ \pi) = l^s(p,\ \pi) + \pi = (l^s(p,\ \pi) - l^d(p)) + py^s(p).$$

[4] I owe this observation to a remark made by R. GOODWIN on a related paper of W. SEMMLER and myself communicated to me by W. SEMMLER as a proposal for future research.

Due to this law we shall neglect the labor market in the following and will investigate the question of stability of general equilibrium by means of the market for goods. For simplicity, we shall also restrict our considerations to the set of regular economies, i.e., in particular assume that all equilibria of our economy exhibit a regular Jacobian [and are thus finite in number, if the usual boundary conditions on the sign of excess demands are added, cf. Dierker (1974, Ch. 1 and 10) and Kirman (1989) for details].

The above demand function d can obviously be rewritten as a function of the two variables p and l^d and will be denoted by $d(p, l^d)$ for simplicity. According to Mas-Colell (1986, p. 65) we have for the partial derivative of this function $d_p \leq 0$ if and only if the weak axiom of revealed preferences holds true, a situation which we will normally not assume as given in the following. We denote by $l^d = l(y^s)$ the inverse of the production function (i.e., planned employment as a function of planned output) and will abbreviate from now on l^d and y^s by l and y for simplicity. The function $l = l(y)$ thus represents the (minimum) cost function in the present model.

Consider now a given (interior) equilibrium of the above simple model, i.e., a situation of the following type

$$d(p^*, l(y^*)) = y^* > 0 \tag{1}$$

$$l'(y^*) = p^* > 0 \tag{2}$$

where $0 \leq l''(y)$ must hold true in a neighborhood of y^* (locally decreasing or constant returns to scale must prevail at a given equilibrium!). Out of equilibrium y^*, p^* the following type of *tâtonnement process* has been suggested by Mas-Colell (1986) as a formalization of Walras' views on the market dynamics in a production economy:

$$\dot{p} = \alpha \cdot [d(p, l(y)) - y], \quad \alpha = const > 0 \tag{3}$$

$$\dot{y} = \beta \cdot [p - l'(y)], \quad \beta = const > 0 \tag{4}$$

Verbally stated the dynamics (3), (4) says that prices are adjusted according to the excess demand on the market for goods and that supply is adjusted in view of extra profits or losses, i.e., according to the discrepancy between the current price for goods and the marginal wage costs of producing the current supply. Such a process has since long been related with the writings of Walras by a few authors, most notably by M. Morishima (1959, 1977) and R. Goodwin (1953, 1989).

For the *Jacobian* J of (3), (4) we get *at the equilibrium point* p^*, y^*

$$J = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} d_p & -1 \\ 1 & -l'' \end{pmatrix} \tag{5}$$

since $d_y = d_\pi \pi'(l) l'(y) = 0$ at $l^*(y^*)$. Note again, that we will only consider regular equilibria ($\det J \neq 0$) in the following. In the case of the weak axiom we therefore have trace $J \leq 0$ (since $l''(y^*) \geq 0$) and det $J > 0$ and thus get *local asymptotic stability* if either $d_p < 0$ or $l''(y) > 0$ holds true in addition, i.e., in case of a negatively sloped $d(\cdot, y)$-curve or for strictly increasing marginal costs.

The following Figure 1 provides a simple illustration of the complicated dynamic behavior that can be expected even for the above simple one-input one-output economy — due to the vastly arbitrary nature of excess demand functions in models of general equilibrium. This picture seems to suggest that not much can be gained in such a general equilibrium setup from using a cross-dual dynamics à la Walras instead of the standard one-sided pure price dynamics generally used for such systems.

However, important similarities in the formulation of Walras' cross-dual dynamics and the dynamic processes as they were formulated by the Classics and Marx [Their 'tendency of profit-rates to equalize', cf. Flaschel/Semmler (1987) for details] suggest, on the one hand, that this dynamics has a much wider economic background and plausibility than its modern one-sided counterpart, the so-called law of demand. And, on the other hand, the long tradition that this latter process has in economic theorizing opens up the possibility that further plausible modifications of it can be found — by a careful reflection of the features that such a classical dynamics may exhibit — which will improve its stability properties. A

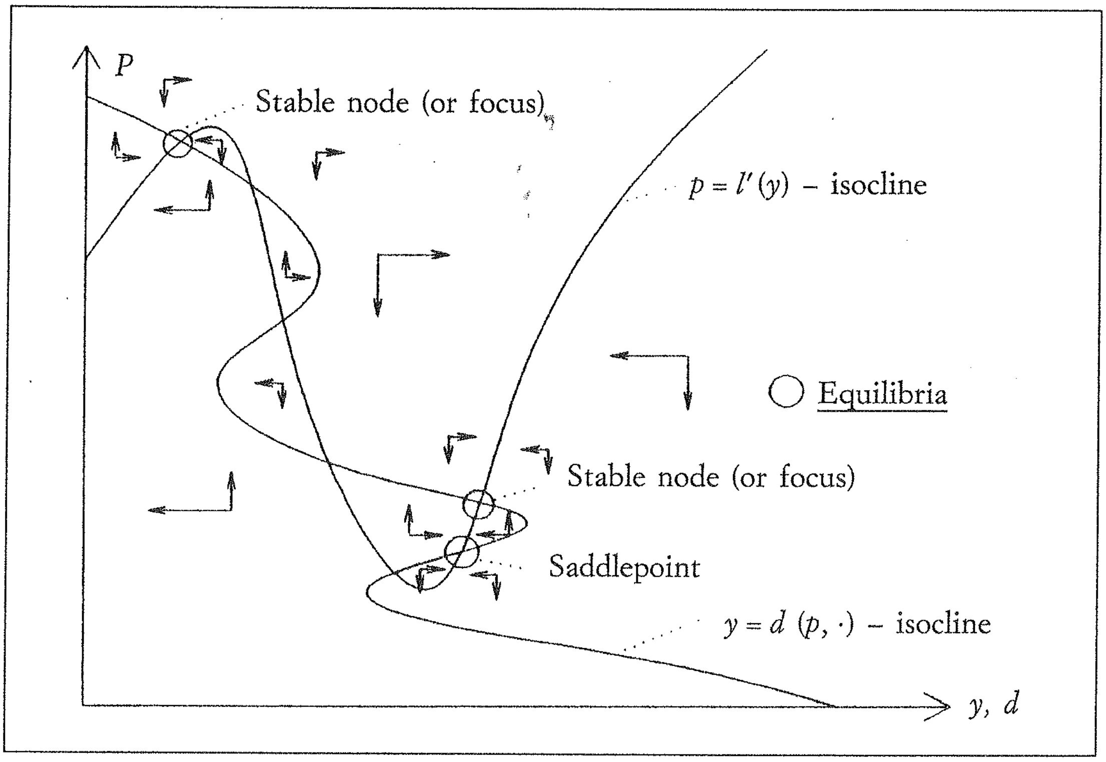

Fig. 1: Cross-dual dynamics in the one-input one-output case.

suggestion of this kind is the simple idea that rising profit-rate differentials will exercise a stronger influence on the conditions of supply than falling ones, cf. again Flaschel/Semmler (1987) in this regard. This idea has been applied in Flaschel (1991) in an extended form to the Walrasian dynamics here under consideration and will be the starting point of our following study of the global properties of such a dynamics. Our following extension thus integrates the effects caused by the direction and the rate of change of excess-profitability with those that are generated by excess-demands. It thereby leads to the following revision of the dynamics (3), (4):

$$\dot{p} = \alpha \cdot [(d(p,\ l(y)) - y) + \gamma_1 \cdot \dot{\overbrace{(d(p,\ l(y)) - y)}}] \qquad (6)$$

$$\dot{y} = \beta \cdot \overbrace{[(p - l'(y)) + \gamma_2 \cdot (p - l'(y))]}^{\cdot} \qquad [7)$$

In this new dynamics, prices and quantities do not only react with respect to the *level* of excess demand and the excess of prices over costs, but also with regard to their time rates of change, i.e., rising discrepancies or disequilibria excercise a different influence than falling ones on the rates of change of prices and quantities. Note here, that the points of rest of this new dynamics are the same as in the original dynamics (3), (4).

This fairly natural extension of the dynamics (3), (4) now in particular makes it possible to formulate the following two proposition — if we in addition assume that $\gamma_1 = \gamma_2$ holds true throughout [see Flaschel (1991) for details]:

Proposition 1 *Consider an interior equilibrium of the system (6), (7). There exists* $\gamma_0 \geq 0$ *such that* (p^*, y^*) will be locally asymptotically stable for all $\gamma \geq \gamma_0$ *with regard to the given adjustment process if* $\gamma = \gamma_1 = \gamma_2$ *is assumed throughout.*

The above proposition states that market pressures — in combination with price/cost differentials — can be reformulated in such a way that the stability of all economic equilibria of a given regular economy will come about. Yet, this proposition also shows that the information which 'markets' need in order to allow for such generally stable adjustment processes toward equilibrium considerably exceeds the information that they have to provide — on a theoretical level — for the derivation of the existence of such equilibria.

Making use of the reformulation (9) of the process (6), (7) one can furthermore show:

Proposition 2 *The adjustment process (6), (7) will give rise to a locally stable generalized Newton method*[5] *if its parameter* γ *is chosen as a function of the Jacobian of this dynamics in an appropriate way.*

[5] See Flaschel (1991) for details.

Roughly speaking this says that the new dynamics will lead us locally toward the equilibria for all economies in the 'neighborhood' of a given one — if the parameter γ of the derivative feedback mechanism is fixed appropriately — and that this mechanism provides an economic example for the formal discussion of generalized Newton Methods and their properties [see again Flaschel (1991) for details].

The aim of the present paper, however, is to treat global questions. To this end, let us first note that cross-duality — in its most basic form [here $d_p = l'' = 0$] — typically gives rise to purely cyclical movements [cf. also Flaschel/Semmler (1987) in this regard], since its Jacobian in such cases is of purely skew-symmetric type. Its (linearized) dynamics consequently is of the same neutral type of stability which characterizes the well-known growth cycle model of R.Goodwin (1967). Taken by itself it thus neither gives reason to expect global stability nor total instability for the dynamics it implies. Instead, it, in fact depends on the particular type of the additional forces which decide whether such a cycle model will become globally stable or even globally asymptotically stable in the end.

Such a statement can be illustrated with regard to the dynamics (3), (4) by means of figure 1 in a straightforward way:

- The sign of d_p at the equilibrium point p^*,y^* first of all decides whether this point is locally stable ($d_p < 0$) or not [assuming $l'' < 0$ for simplicity].
- In the case $d_p > 0$ local asymptotic stability depends on the relative size of the adustment parameters α,β.
- In the case $d_p < 0$ local asymptotic stability may even be of purely monotonic kind (non-cyclical) if the terms in the diagonal of (5) are chosen sufficiently large.
- Whatever the particular type of local dynamics that prevails at the various equilibria of figure 1, it can easily be seen (graphically[6]) that a system like that of figure 1 must be globally stable (or limited) — if the dynamics on the boundary of $\mathbb{R}^2_+$ is redefined in a natural way

Considering the above figure 2 we can see that it is mainly due to the outward bounds assumed with respect to the demand function that the considered cross-dual dynamics must be stable in the large. Therefore, the forces which delimit the system's behavior have nothing to do with the cross-field type of adjustment assumed to characterize the price/quantity movements around equilibria. In addition, figure 3 in Mas-Colell (1986, p. 65) — with its unique equilibrium, cf. our figure 5 below — and the comments which accompany it very nicely indicate how the details of the situation given by our figure 2 can be further analyzed, firstly by a mixture

[6] A formal proof would have to reformulate the Poincaré-Bendixon Theorem appropriately to take account of some particular features of figure 2.

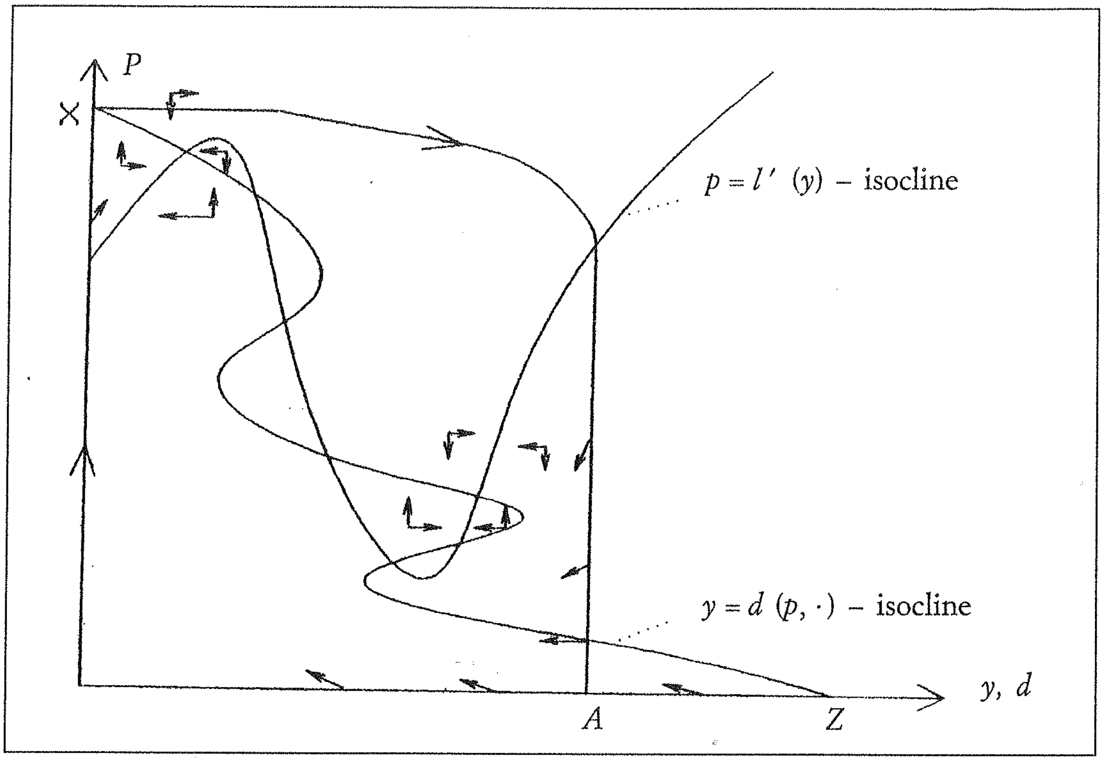

Fig. 2: Cross-dual dynamics and global stability.

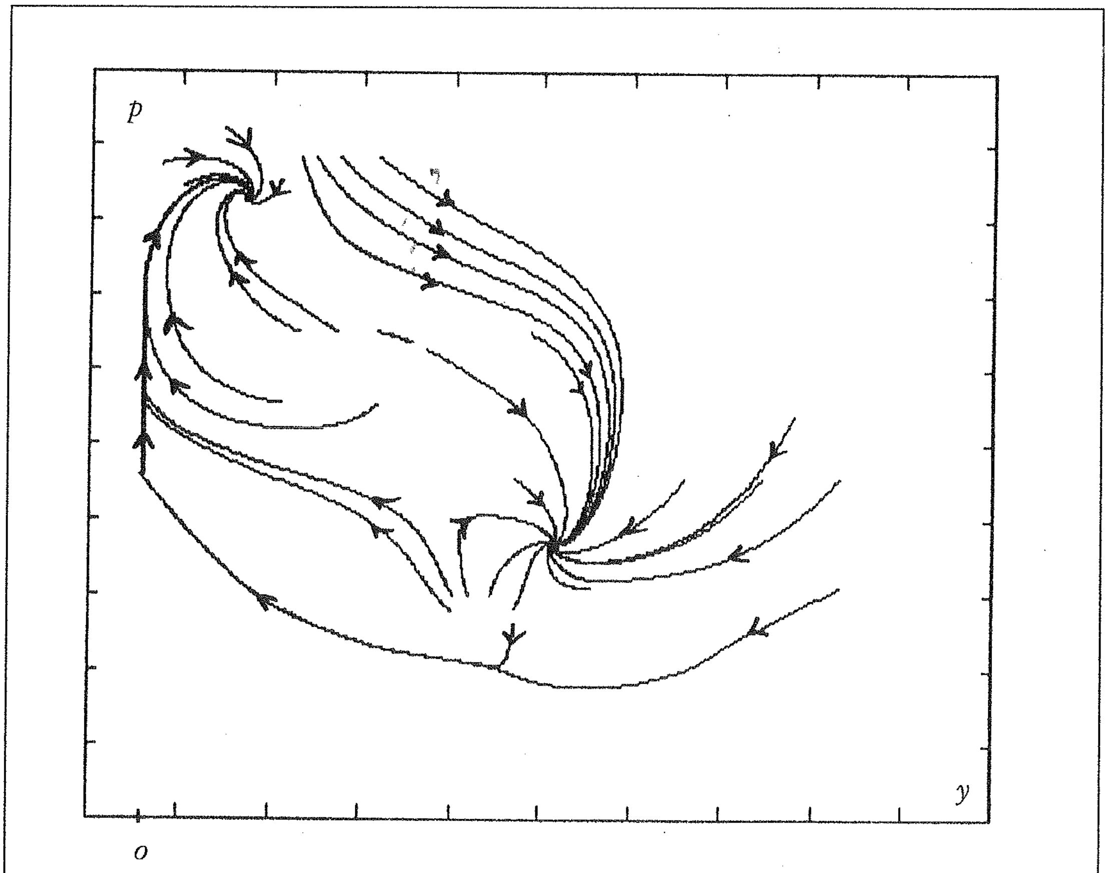

Fig. 3: A simulation study of figures 1, 2.

of local and global considerations (deriving global limit cycles by means of the Poincaré-Bendixon Theorem), and secondly by purely local considerations (where one may derive in addition to the existence of the globally stable limit cycle of the first kind, e.g., unstable limit cycles near the equilibrium by means of the Hopf bifurcation theorem). We do not want, however, to go into the many details of such an analysis here. Instead, we close this section by a simulation study based on the dynamic situation depicted in figure 1[7] This simulation roughly indicates the regions where the two point attractors of figures 1,2 have their basin of attraction and it also shows — on its left hand side — which of the trajectories may hit the boundary of the positive orthant (where the dynamics is governed by one law only).[8]

3. DERIVATIVE CONTROL AND GLOBAL STABILITY

One great disadvantage in the arguments put forth so far to support the global stability of our cross-dual dynamics (3), (4) is that they rely on side conditions where a controlled economic behavior is not really to be expected, i.e., on boundary conditions such as $y = 0$ or $p = 0$. These choices are simply too extreme to be convincing that the true forces that keep the system within economically sensible bounds have thereby been detected. It is in this regard that derivative forces like those in (6) and (7), which depend on the direction and the amount of change of disequilibrium situations, may become important as we shall attempt to show in the following.

In compact form the system (6), (7) can be reformulated as follows:

$$\dot{z} = F(z) + <\gamma> \dot{F}(z) \tag{8}$$

where z stands for $(p,y)'$ and where $<\gamma>$ is the diagonal matrix containing the adjustment parameters γ_1, γ_2 in its diagonal. These parameters are both assumed to the positive but not necessarily equal to each other in the following.[9] The above reformulation of system (6), (7) in turn implies

$$\dot{z} = F(z) + <\gamma> F'(z) \cdot \dot{z}, \text{ i.e.}$$
$$\dot{z} = (I - <\gamma> F'(z))^{-1} F(z) \tag{9}$$

as long as the matrix $I - <\gamma> F'(z)$ is regular.

We have shown in Flaschel (1991) that a choice of a sufficiently large common parameter $\gamma_1 = \gamma_2$ will make all points of rest of such a system

[7] With $\beta = 1$ for simplicity.

[8] The lower left-hand corner of the rectangle which surrounds the depicted trajectories has as coordinates the values (-.7,-.7) [and (11,9.6) on the upper right hand side].

[9] Note that we could only derive special, *i.e.*, two-dimensional results in Flaschel (1991) for the cases where either γ_1 or $\gamma_2 = 0$ was assumed to hold true.

locally asymptotically stable. We consequently know [by index considerations, cf. Dierker (1974), for example] that our system (9) cannot be well defined over the whole range that is considered in figure 2. A general investigation of the global properties of (9) may therefore be quite complicated and demanding.

Hence, to simplify our considerations, let us consider a regular equilibrium z^* and an open and connected domain D containing z^* as a *unique equilibrium* where the dynamics (9) is well defined and where the function

$$G(z) = z - z^* - <\gamma> F(z) \text{ is } C^2$$

and has a regular Jacobian at all points of the domain D.
In order to study the dynamics (9) on an appropriately chosen part of such a domain the following auxiliary function V will be of help

$$V(z) = \|G(z)\|^2 = \|z - z^* - <\gamma> F(z)\|^2 \qquad (10)$$

where $\|\cdot\|$ denotes the Euclidean distance (derived from the scalar product $\ll .,. \gg$).

Proposition 3 *The function V is a strict Liapunov function at* z^*[10] *if the adjustment parameters in* $<\gamma>$ *are chosen sufficiently large.*

Sketch of proof: By the definition of V we have $V(z^*) = 0$ and because of the regularity of the function $G(z) = z - z^* - <\gamma> F(z)$ at z^* we also know that $V(z) > 0$ must hold true in $U - z^*$ for a suitably chosen neighborhood U of the equilibrium z^*. According to Liapunov's stability theorem[11] it remains to be shown that the condition $\dot{V} < 0$ also holds in such a neighborhood $U - z^*$ of the given equilibrium. Differentiating V along the trajectories of (9) gives [Cf. Dieudonné (1960, p. 144)]:

$$\begin{aligned} \dot{V} &= 2 \ll (I - <\gamma> F'(z))\, \dot{z},\ z - z^* - <\gamma> F(z) \gg \\ &= 2 \ll F(z),\ z - z^* - <\gamma> F(z) \gg \\ &= 2[\ll F(z),\ z - z^* \gg - \ll F(z), <\gamma> F(z) \gg] \end{aligned}$$

Now, since z^* is a regular equilibrium of F, we can apply the mean value theorem [cf. Dieudonné (1960, p. 155)] with regard to the function F^{-1} in order to get that

$$\|F^{-1}(q) - F^{-1}(0)\| \leq c \cdot \|q - 0\| \quad \text{or}$$
$$\|z - z^*\| \leq c \cdot \|F(z)\|$$

must hold true for a suitably chosen neighborhood U' of z^* and a positive constant c.

[10] Cf. Hirsch/Smale (1974, p. 193) for the definition of this concept.
[11] Cf. again Hirsch/Smale (1974, p. 193).

This gives

$$V \leq 2[||F(z)|| \cdot ||z - z^*|| - \gamma_{\min} \cdot ||F(z)||^2],$$
$$\leq 2[||F(z)||^2 \cdot c - \gamma_{\min} \cdot ||F(z)||^2],$$
$$= 2(c - \gamma_{\min})||F(z)||^2$$

where $\gamma_{\min} > 0(!)$ is the minimal parameter in the set of all adjustment parameters γ_i. The choice $\gamma_{\min} > c$ then immediately implies that $\dot{V} < 0$ must hold true in $U' - z^*$.

Proposition 4 *The dynamics (9) is globally asymptotically stable on the largest set* $V^{-1}([0, a]), a > 0$ *where the function* G is of the assumed type[12] *and where* $\dot{V}(z) < 0$ *holds true for* $z \neq z^*$.

Proof: See Theorem 2 in Hirsch/Smale (1974, p. 196).

Assumptions:

Assume that there exist a positive real number such that the closed set $K = V^{-1}([0,a])$ is compact and contained in the domain D. Let us denote the maximum of $|z_i - z_i^*|$ for $z \in K$ by k_i, $i = 1,2$.

We already know that there exist numbers ϵ_1, ϵ_2 — positive and sufficiently small — such that the mapping F maps an open neighborhood U^ϵ of the equilibrium z^* one-to-one onto the open set that is determined by $|F(z)_i| < \epsilon_i, i = 1,2$. Since z^* is the only equilibrium of (9) in the compact set K we can choose these ϵ_i in such a way that

$$|F(z)_i| \geq \epsilon_i \quad \text{for all} \quad z \in K - U^\epsilon, \quad i = 1,2 \tag{11}$$

will hold true in addition.

On the basis of these assumptions we can show:

Proposition 5 *The function* V *is a non-local Liapunov function around the equilibrium* z^*, *i.e.,* $\dot{V} < 0$ *will hold true in the set* $K - U^\epsilon$ *if the adjustment parameters* γ_i *are larger than* k_i/ϵ_i, $i = 1,2$.

Sketch of proof: Explicitly referring to coordinates we get for $\dot{V}$:

$$\dot{V} = 2[\sum_i F(z)_i(z_i - z_i^*) - \sum_i \gamma_i (F(z)_i)^2]$$
$$\leq 2[\sum_i |F(z)_i||z_i - z_i^*| - \sum_i \gamma_i (F(z)_i)^2]$$
$$\leq 2[\sum_i (|F(z)_i||k_i| - \gamma_i|F(z)_i|\epsilon] \qquad < 0$$

if $\gamma_i > k_i/\epsilon_i$ holds true.

[12] Cf. our above description.

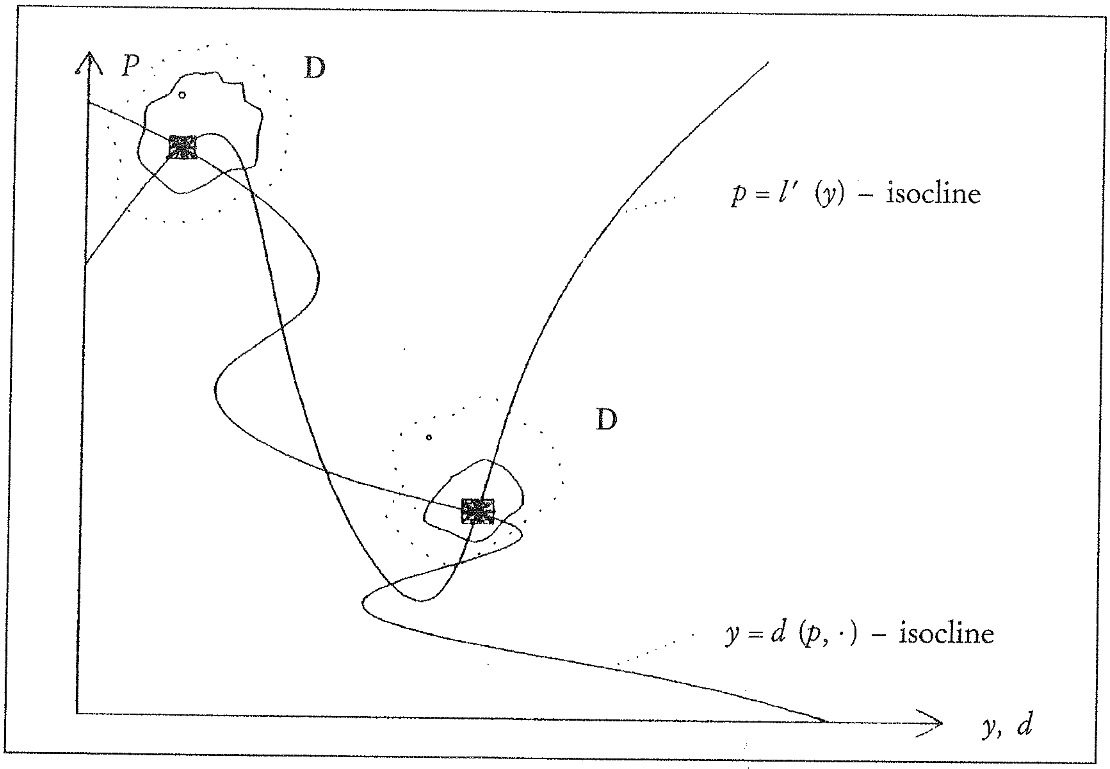

Fig. 4: Basins of attraction of the dynamics (9).

The following figure illustrates the situation we have just considered. Of course, the above estimation is very crude, so that $\dot{V} < 0$ may generally be expected to hold true for much smaller γ_i than have been used to prove the above proposition. Verbally stated this proposition says[13] that all trajectories of the dynamics (9) which start in K will reach U^ϵ after some finite time [from where on their further behavior is no longer obvious since z^* need not be an asymptotically stable equilibrium of (9)]. The "limit set" of the "invariant set" K, therefore, must be contained in U^ϵ, but may have quite a complicated structure.

We conclude that the dynamics (9) still allows a variety of possibilities to tailor it to the particular equilibrium under consideration. In particular, one may also assume that the parameters γ_i will depend on the discrepancies in supply and demand as well as on prices and marginal costs — or alternatively on the distance that actual prices and quantities $z = (p, y)'$ will have from their equilibrium values z^* — for example, in the following way

- $\gamma_i \equiv 0$ in a neighborhood of z^*, *i.e.*, derivative forces only come into being if price-cost or supply-demand differences become sufficiently pronounced,

[13] We know, however, of no reference where such a non-local type of Liapunov function is considered and where an exact proof for the mathematical part of this proposition is given.

- $\gamma_i = const$ large enough to ensure $\dot{V} < 0$ on the boundary of $K = V^{-1}[0, a]$, *i.e.*, on $V^{-1}(a)$.
- $\gamma(z)$ continuous.

Such a situation would then imply that K is positively invariant, *i.e.* the dynamics (9) must then be globally stable with respect to this domain *but it does not imply anything further as to the type of behavior that this derivatively controlled system will have in the interior of K*!

Such a result definitely leaves the realm of neoclassical stability analysis (according to which the 'world' consists of point attractors only). Instead, it approaches Marxian ideas on the capability of capitalist economies to 'reproduce' themselves, *i.e.*, to remove certain 'obstacles' which threaten their continuing existence if these 'obstacles' (here: discrepancies in profitability and in quantities supplied and demanded) reach certain thresholds. Of course, the present model is still very far away from such a speculative picture of the working of a capitalist economy, *e.g.*, because 'profitability' is not yet treated in an adequate way — and in particular also because of our reliance on tâtonnement adjustment procedures.

In his brief comments on global stability Mas-Colell (1986, pp. 63ff.) states that one may perhaps "attach some meaning to a globally convergent dynamics but a limit cycle, say, lacks any real significance." Our opinion with regard to this question is that tâtonnement dynamics has no real significance at all if the word 'real' is taken literally. Instead, it is a methodological device that may be useful for comparing alternative ideas or proposals on the types of adjustments conceived to take place in a capitalist economy — in some sort of a vacuum, where many complicated feedbacks of a consistently formulated economic environment are set aside and where one can test in a first round the effectiveness of such proposals with regard to their basic dynamic properties. To lay bare the essentials of an economic adjustment mechanism in an artificial environment may thus represent one useful approach on the way in constructing proper types of adjustment processes for various types of capitalist market economies — as the global stability results we have reached above, for example, should demonstrate.

4. SIMULATION STUDIES

In order to allow for some basic simulation studies of the effects of derivative control by means of an explicit treatment of the implied dynamic system (9) we shall further simplify the situation we have used for illustrative purposes in section 2. We shall now consider the example that is depicted in Mas-Colell (1986, p. 65), here by making use of the following explicit

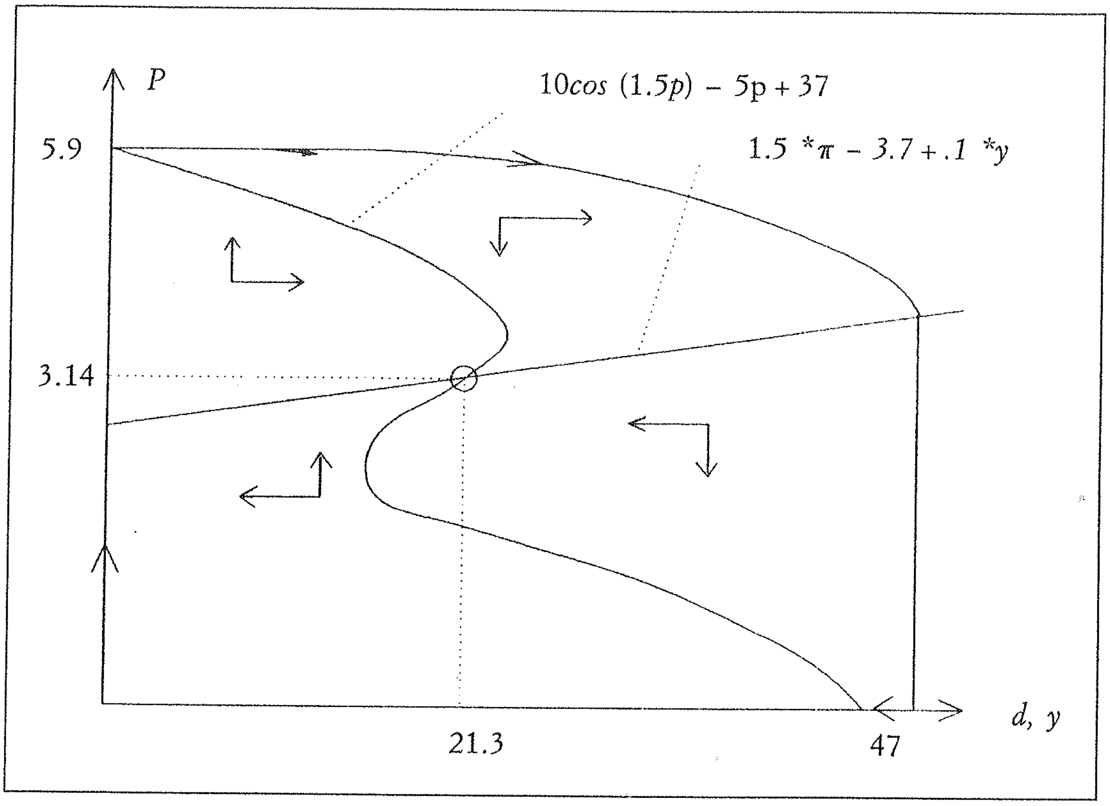

Fig. 5: Global stability for unique and instable equilibria.

form of his demand function $d(p)$ — which qualitatively is of the same shape as the one used by Mas-Colell in his figure 3:[14]

$$d(p) = 10\cos(1.5p) - 5p + 37 \tag{12}$$

We furthermore assume a marginal cost curve of the type $l''(y) = c_1 + c_2 \cdot y$ for simplicity. This choice gives rise to the following type of phase portrait:[15]

In this case we get for the cross-dual dynamics (3), (4), *i.e.*, for $<\gamma> = 0$, the equations

$$F(z) = \begin{pmatrix} \alpha[10\cos(1.5p) - 5p + 37 - y] \\ \beta[p - c_1 - c_2 y] \end{pmatrix} \tag{13}$$

and therefore for the matrix that is involved in the formulation of the dynamics (9):

$$I - <\gamma> F'(z) = \begin{pmatrix} 1 + \alpha\gamma_1[15\sin(1.5p) + 5] & \alpha\gamma_1 \\ -\gamma_2\beta & 1 + \beta c_2 \end{pmatrix}. \tag{14}$$

[14] See also assumption (ii) on page 64.

[15] We shall make use of $c_2 = .025, c_1 = 2.61$, *i.e.*, $p^* = \pi$, $y^* = 21.3$ and $\alpha = 1$, $\beta = 1$ in our following computer simulations.

This finally gives for the dynamics (9) the following system of ordinary differential equations of dimension 2:

$$\dot{z} = (\dot{p}, \dot{y})' = \frac{1}{(1+\alpha\gamma_1[15\sin(1.5p)+5])(1+\beta d)+\gamma_1\gamma_2\alpha\beta} \cdot \begin{pmatrix} (1+\beta d)\alpha[10\cos(1.5p)-5p+37-y]-\alpha\gamma_1\beta[p-c_1-c_2y] \\ \gamma_2\beta\alpha[10\cos(1.5p)-5p+37-y]+(1+\alpha\gamma_1[15\sin(1.5p)+5])\beta[p-c_1-c_2y] \end{pmatrix}$$

This is the dynamic system that we shall briefly study by means of computer simulations in the remainder of this section.

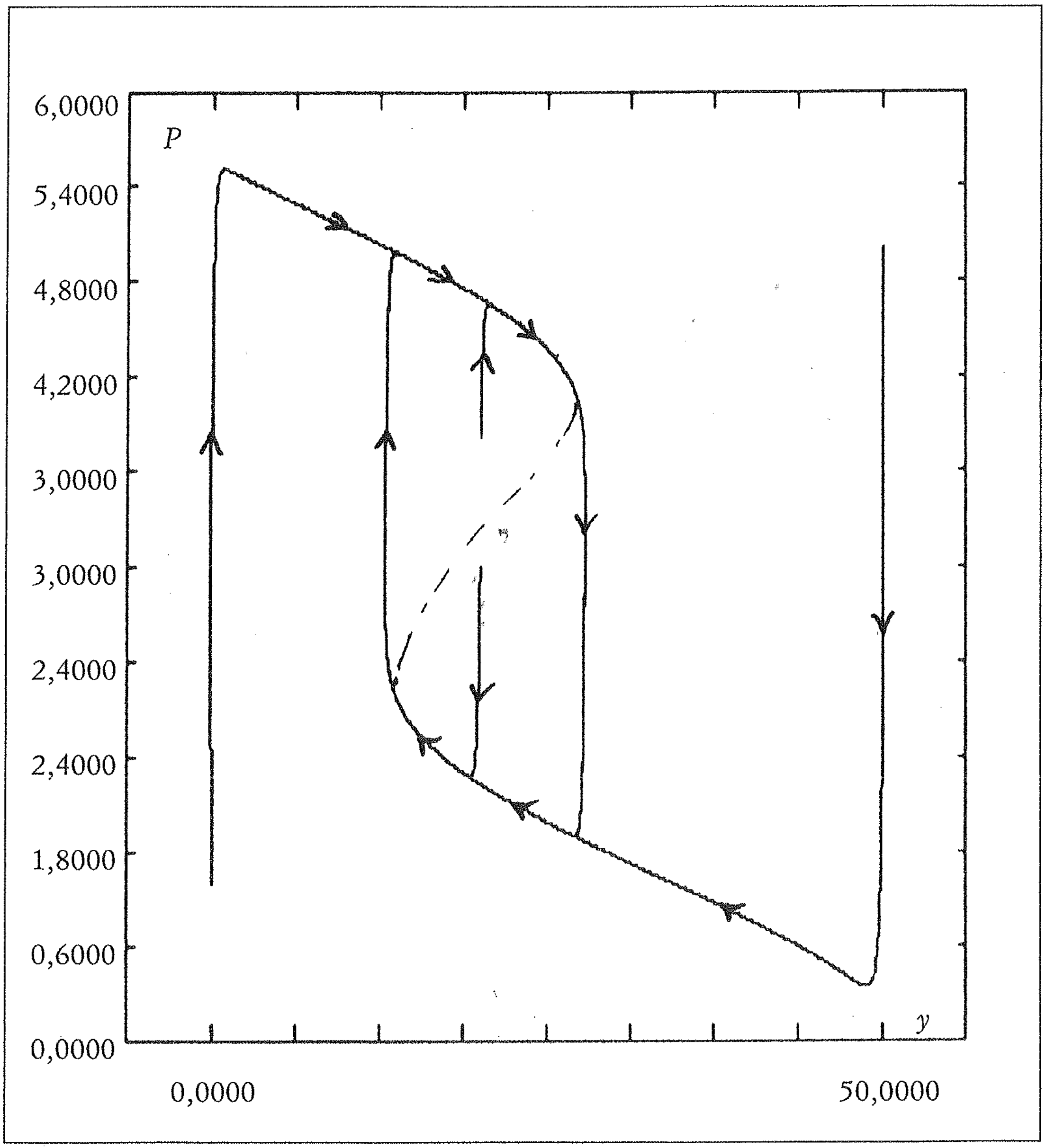

Fig. 6: No derivative control: the fold catastrophe.

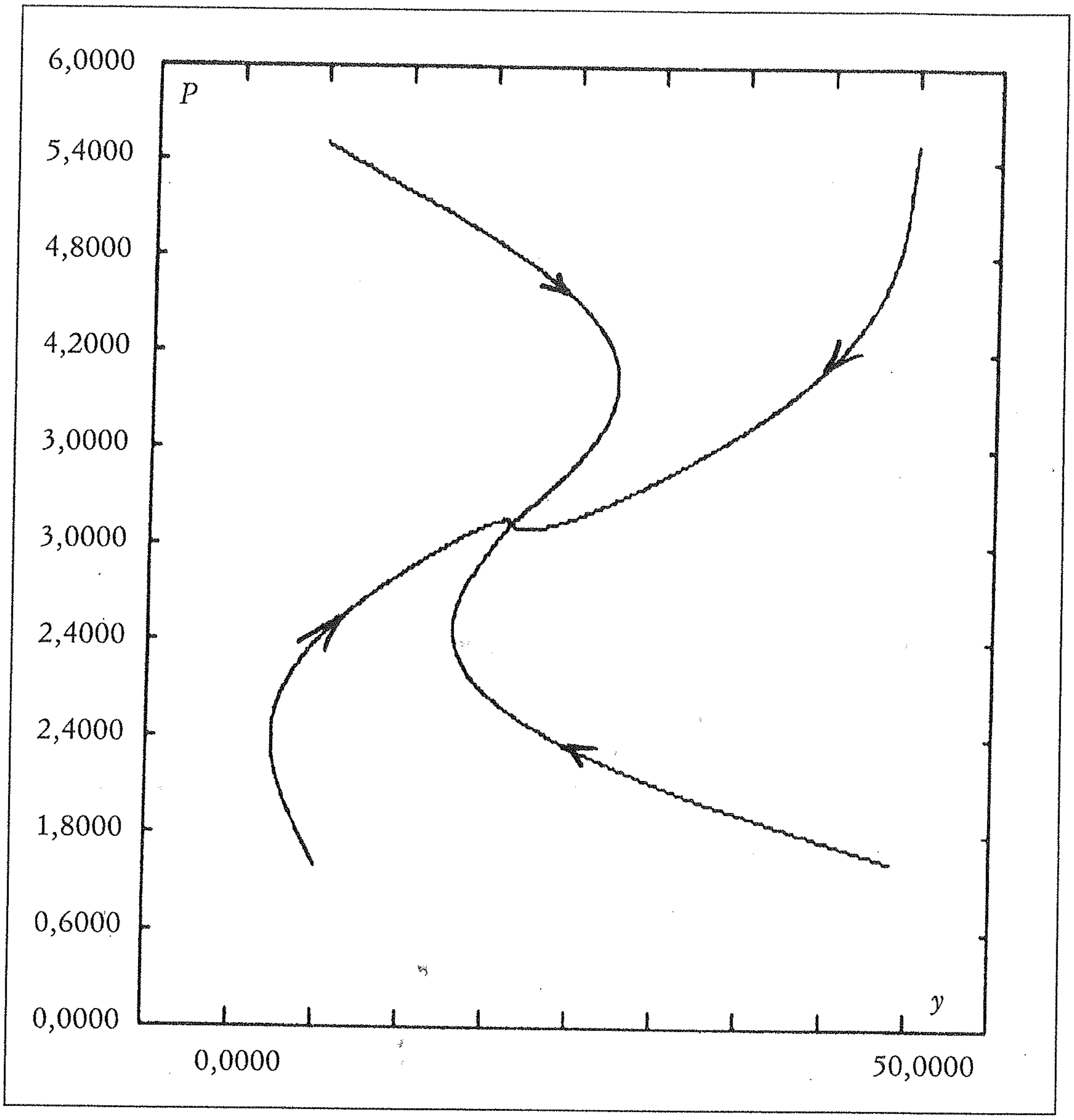

Fig. 7: Complete derivative control: global asymptotic stability.

Let us first consider the case of no derivative control ($\gamma_{1,2} = 0$). The above example then gives rise to a very fast price dynamics. Up to pretty short traverses we consequently always have that the movement is very near to the $\hat{p} = 0$ isocline where the goods market is in equilibrium. Yet, due to price/cost differentials, output keeps expanding (on the upper part of this isocline) and contracting (on its lower part) until it reaches the critical point where the demand curve bends backwards (becomes positively sloped). The possibility for an upper, respectively lower, goods-market equilibrium then disappears and there follows the well-known rapid downturn, respectively upturn, here in terms of goods-market prices, of such a dynamics which again restores approximate goods-market equilibrium on the lower/upper part of the demand curve.

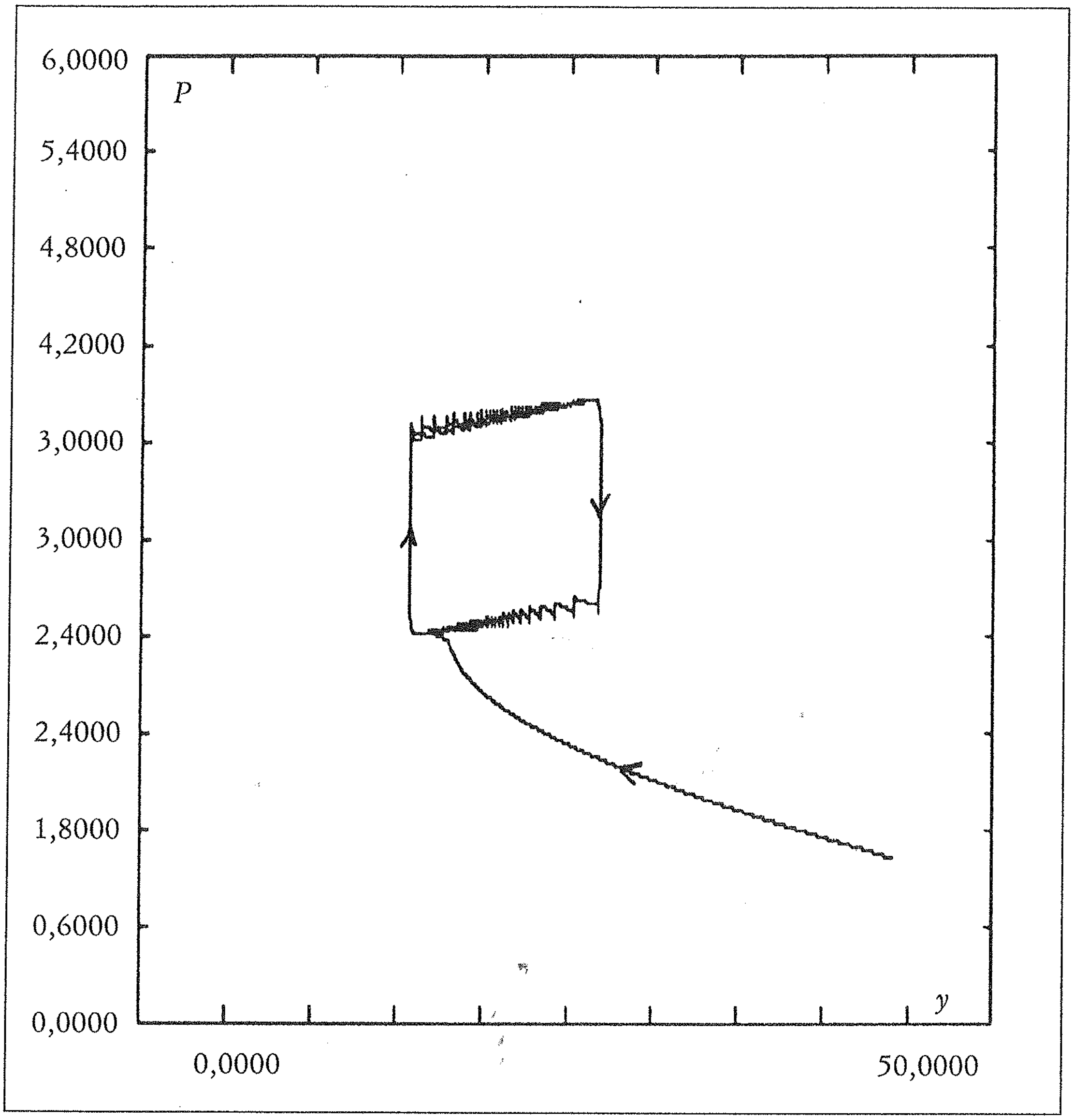

Fig. 8: Partial derivative control: Cyclical viability.

The following simulation of this behavior has been conducted by making use of a step length of 0.01. Increasing this step length to, *e.g.*, 0.1 leads — in addition to the results shown in figure 6 — to overshooting and damped cycles around the $\dot{p} = 0$ isocline when the system moves from its upper part to its lower (and v.v.).

The initial situation is therefore characterized by a very stable type of limit cycle where one of the isoclines governs nearly all of its shape. Can this pronounced cyclical dynamics be overcome through the introduction of derivative control — by making its influence sufficiently strong? The following simulation drastically demonstrates that this can indeed be the case [the parameters $\gamma_{1,2}$ have both been set equal to 20 to obtain this figure]. Derivative control thus not only leads to a damped cyclical movement, but is also capable of removing the cyclical nature of cross dual dynamics in a very radical fashion.

Finally, in figure 8 — following a recommendation of R. Goodwin — the same strong influence of a derivative control $\gamma_{1,2} = 20$ is combined with the ordinary cross-dual dynamics — in a bang-bang fashion, *i.e.*, the dynamics (3), (4) applies as long as excess demand is less than 2 in absolute value and excess price is less than 0.6 in absolute value.

In such a case the cycle is, of course, no longer removed, but only becomes squeezed when the dynamics reaches the domain where the derivative forces become operative. This demonstrates that the inclusion of derivative control mechanisms — which are highly plausible whenever existing imbalances cross certain thresholds — by no means must give rise to the 'world of point attractors' of ordinary neoclassical economics.

Note finally, that we have made use here of the possible combinations of cross dual and derivative control mechanisms only in a fairly prelliminary way. Many further results may be expected for other types of combinations, different parameter sizes,, and in particular for different types of economic environments. Also, the possibility of only a partial derivative control (where some of the adjustment parameters are set equal to zero, for example by excluding the law of demand, but not the law of profitability from the derivative control mechanism) should be investigated in much more detail than was done in Semmler/Flaschel (1987) and Flaschel (1991). This, however, will demand different strategies of proof if global stability results are to be obtained — as is obvious from the proofs given in section 3. Such questions must however be left for future investigations.

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References

Beckmann, M. J. and Ryder, H. E. (1969): "Simultaneous price and quantity adjustment in a single market" *Econometrica*, vol. 37, pp. 470-48.

Dierker, E. (1974): *Topological Methods in Walrasian Economics*. Heidelberg, New York: Springer.

Dieudonné, J. (1969): *Foundations of Modern Analysis*. New York: Academic Press.

Flaschel, P. (1991): "Stability — independent of economic structure? A prototype analysis". To appear in: *Structural Change and Economic Dynamics*.

Flaschel, P. and Semmler, W. (1987): "Classical and neoclassical competitive adjustment processes". *The Manchester School*, vol. 55, pp. 13-37.

Goodwin, R. M. (1953): "Static and dynamic general equilibrium models". In: R. M. Goodwin: *Essays in Linear Economic Structures*. London: Macmillan.

Goodwin, R. M. (1967): "A growth cycle". In: C. H. Feinstein (ed.), *Socialism, Capitalism, and Economic Growth*. London: Cambridge University Press.

Goodwin, R. M. (1989): *Essays in Nonlinear Economic Dynamics*. In: P. Flaschel and M. Krüger (eds.), Dynamische Wirtschaftstheorie, vol. 3, Bern: Peter Lang.

Hirsch, M. W. and S. Smale (1974): *Differential Equations, Dynamic Systems, and Linear Algebra*. New York: Academic Press.

Kirman, A. (1989): "The intrinsic limits of modern economic theory: The emperor has no clothes". *Economic Journal*, vol. 99 (Conference Volume), pp. 126-139.

Mas-Colell, A. (1986): "Notes on price and quantity tâtonnement dynamics". In: H. Sonnenschen (ed.): *Models of Economic Dynamics*. Heidelberg, New York: Springer, pp. 49-68.

Morishima, M. (1959): "A reconsideration of the Walras-Cassel-Leontief models of general equilirbium". In: K. Arrow et al. (eds.): *Mathematical Social Sciences*. Stanford, Stanford University Press.

Morishima, M. (1977): *Walras' Economics*. Cambridge: Cambridge University Press.