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Abstract

Most studies of the frequency of reswitching and reverse capital deepening were based on two or three sector models and came to the conclusion that these phenomena are rare. Here it is shown that the probability of isolated reswitching tends to zero in large economic systems. The assumptions of the new mathematical theorems proposed are supported by an empirical enquiry. The randomness of input-output systems helps to explain the result. If the number of switch-points tends to infinity in large systems, it is not certain that even one represents a case of reverse capital deepening. The focus of the critique of capital will have to change.

Keywords: capital theory, reswitching, reverse capital deepening, Sraffian economics, input-output analysis
JEL codes: B24, C62, C67, D57

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1 Introduction

Reswitching and reverse capital deepening were the subject of what was perhaps the most heated debate in economics in The Quarterly Journal in the 1960s (Samuelson, 1966). While reswitching and reverse capital deepening are incompatible with a pure form of neoclassical theory, modern mainstream neoclassicals regard these phenomena as exceptional paradoxes and ignore them in consequence. These paradoxes will be explained formally in Section 3. Readers not familiar with the definitions should pick them up there. Here we start with an overview. It is the purpose of this paper to confirm the exceptional character of the paradoxes for large systems. We want to show that the probability of their occurrence tends to zero as the number of the sectors of the system tends to infinity. I do not think that this saves neoclassical theory from critique; I rather believe that a modified critique will result, but this is not the subject of this paper. After the general introduction in Section 1, Section 2 provides the background of the theory of prices and focuses especially on the degree to which relative prices change with distribution. Section 3 analyses where techniques leading to reswitching are located in the set of potential techniques. Section 4 illustrates the geometrical problems graphically, Section 5 provides an empirical investigation on the basis of input-output analysis and Section 6 contains the theorems supporting the general contention. Some conclusions follow in Section 7, where we ask how probable it is that at least one case of reverse capital deepening can be found in a large system.

Reswitching appears as a possibility in an Austrian model already in an exchange between Böhm-Bawerk and Irving Fisher (Fisher, 1907, p. 352; Schefold, 2017, pp. 224, 270, 212). Sraffa's (1960) example of reswitching is formally close to the one proposed by Fisher, but aims at a more fundamental critique; it uses non-basics in a basic system. Levhari (1965) thought that reswitching could be excluded in basic systems, but he was mistaken as several authors showed by means of counterexamples; Samuelson (1966) summarised the debate, adding an Austrian example that in turn reproduced the structure of the example proposed by Sraffa. In consequence, a number of authors have since taken up the challenge to determine whether reswitching is exceptional or sufficiently frequent to question the realism and the applicability of neoclassical theory. If reswitching or, rather, reverse capital deepening were as frequent as the (for neoclassicals) normal case, lowering the wage in the face of unemployment would lead to an uncertain result, for, always according to neoclassical logic, the existing stock of capital would be transformed and, with lower wages, the installation of more labour-intensive techniques would be expected, but just as many labour-saving techniques would appear, with compensating effects, so the employment effect would be uncertain and other parts of the theory would also be hampered. Hence it matters whether reswitching is frequent or rare.

Most authors have examined the question using two- or three-sector models. Austrian variants have been used by Samuelson (1966), as stated, by Hicks (1973), and Laing (1991), and the Austrian model is also discussed in Mainwaring and Steedman (2000). They all conclude that the probability of reswitching or reverse capital deepening is small.

A somewhat different picture is obtained with the Samuelson model of Samuelson (1962), also called corn-tractor model. Here always the same consumption good is produced by a capital good and labour, and, with labour, this capital good reproduces itself in a second process. Technical change can then be illustrated by the familiar example: one replaces the spade by the horse-drawn plough and then this again by the tractor-drawn plough. The transitions are not modelled. Mainwaring and Steedman (2000), D'Ippolito (1987) and Eltis (1973) with a somewhat similar approach estimate the probability for reswitching to be low, while Petri (2011) is alone in believing that it is much higher. It is always a question of identifying a region in the space of the parameter values where reswitching occurs versus a region, where at least one switch results; the probability then is given as the ratio between the corresponding areas. Or, similarly, it is asked whether a given switch occurs with parameters such that the intensity of capital increases with an increasing rate of profit at that switch. The in principle same approach is used when Sraffa models are considered. Woods (1988) analyses a two-sector Sraffa model, and again Mainwaring and Steedman (2000), still concluding that reswitching is not frequent.

The restriction to small models in all the cases considered so far is problematic, if one has an economic theory in view that should be capable of applications. Reverse capital deepening occurs in a specific sector, but it has a macroeconomic implication. The phenomenon may disappear because of aggregation – most obviously, if one aggregates to a one sector-sector model, in which reverse capital deepening is not possible. It is not clear what reverse capital deepening means in a two-sector model or, more generally, one in which aggregation is so high that individual method changes cannot be identified. Technical change would then result in a gradual, near continuous change of coefficients. The logic of the argument seems to require a representation at an intermediate level of aggregation, such as is represented in input-output tables with significantly more than two sectors. One usually assumes homogeneous commodities in pure theory. Sraffa (1960) speaks of "wheat", "iron", "pigs", "gold", but none of these commodities is perfectly homogeneous; standards of fineness are needed even for gold, the price of electricity varies according to time and location, etc. Input-output tables have the advantage that they are the result of international cooperation in the definition of sectors; they are the best makeshift we have for analysing inter-industry structures.

So we must turn to large economic systems with many industries. Schefold (1976) proved that the probability for reswitching is positive for basic, regular Sraffa systems. Sraffa systems are regular as defined in Schefold (1971) as, roughly speaking, systems where prices move in all directions, if the rate of profit varies – a system is not regular, for instance, if the labour theory of value holds. I believed at that time that reswitching would actually be frequent, but an attempt to show it empirically with a PhD student of mine, Zonghie Han, using pairs of input-output tables, led to the opposite result (Han and Schefold, 2006). Zambelli (2018), using a new algorithm, has calculated the envelopes of the wage curves derived from the techniques of 30 countries with 31 sectors. His criteria for judging whether the envelopes are compatible with neoclassical premises are in part different from the ones used here, but Kalb (2022) has shown that Zambelli's results show a frequency of reverse capital deepening that is quite similar to that observed by Han and Schefold (2006): less than 2% of the switch-points observed on the envelope exhibit reverse capital deepening. I was not aware until recently of D'Ippolito's attempt of 1989 to show that reverse capital deepening is rare (D'Ippolito, 1989). This will not be discussed in the present paper, because only a short summary of his long unpublished paper with Mario Latorre has been published. According to this description, D'Ippolito and Latorre used a Monte Carlo method, therefore an approach which is quite different from the one pursued here. I hope to be able to consider it in the near future in a special paper.

I had begun to use random matrices as a possible explanation of the near-linearity of wage curves which we shall have to discuss as a preliminary to the discussion of the probability of reswitching in the next section (Section 2). Combining the insights about random matrices with the empirical lesson, I explored the possibility of getting an approximation to Samuelson's (1962) surrogate production function (Schefold, 2013a). At the same time, I tried to understand another curious finding of the paper with Han of 2006: Far fewer wage curves from a given spectrum of techniques would appear on the envelope than we had expected; an estimate in Schefold (2013b) then was that, if there were s techniques in the spectrum, one should expect at most $\ln s$ wage curves on the envelope. A deeper investigation with the mathematician Götz Kersting led to an estimate of $\frac{2}{3}\ln s$, if the maximum wage rates and the maximum rates of profit are distributed according to a uniform distribution with given bounds. In the relevant range of the rate of profit (bounded away from zero and from very high rates of profit) only one or two wage curves appear on the envelope, of almost equal capital-intensity, and this result carries over with slight qualifications to the normal distribution according to theoretical considerations, according to empirical investigations and according to numerical experiments (Kersting and Schefold, 2021). This implies that there is virtually zero substitution between capital and labour among efficient techniques. Another way of stating it is to say that the capital-labour ratio is given independently from distribution. Hence there is no room for the marginal productivity theory of distribution and one feels invited to return to the post-Keynesian theory of distribution (Schefold, 2021a).

Zero substitution thus becomes an argument for a new critique of neoclassical theory, if reswitching is truly exceptional. It is a little surprise that the absence or near-absence of reswitching is also essential for the new critique, in that the formula for the upper bound of the number of wage curves on the envelope, $\ln s$, had been derived first on the assumption that the wage curves are linear. But it turned out that this estimate can be extended to wage curves that are not linear, provided that reswitching is sufficiently rare (Kersting and Schefold, 2021, p. 523).

Our task now is to show that reverse capital deepening and reswitching are rare. More specifically, we want to show that the probability of their occurrence tends to zero, if n, the number of sectors, tends to infinity. For each finite n, this probability is (under the assumptions to be stated) positive, but increasingly small.

A short overview of how we shall proceed: if a regular Sraffa system is given, we can describe the set $M(r_1)$ of potential methods of production that have one switch with the method of production in use, say the first, denoted by $(\mathbf{a}_1, l_1)$, where $\mathbf{a}_1$ is the vector of commodity inputs for the production of a unit of commodity 1 and l_1 the labour input. The set $M(r_1) \cap M(r_2) \cap M(r_3)$ then is, for instance, the set of potential techniques $(\mathbf{a}_0, l_0)$ that have three switch-points in common with $(\mathbf{a}_1, l_1)$, which means the set of potential methods that produce commodity 1 at the same cost as $(\mathbf{a}_1, l_1)$. This set will not be empty, because it contains $(\mathbf{a}_1, l_1)$, but insignificant, because it is of lower dimension than $M(r_1)$. However, $M^\star(r_1)$, the union of all $M(r)$, where $0 < r < R$ and $r \neq r_1$, will be of the same dimension as $M(r_1)$, if the system is regular. $M^\star(r_1)$ is a subset of $M(r_1)$, and the measure of $M^\star(r_1)$, divided by the measure of $M(r_1)$, will be the probability of reswitching.

What we have examined here is what I call an isolated reswitch: we have an isolated system, with n industries and one method per industry, except in one – here the first – industry, where there is one alternative method, which gives rise to a switch and a reswitch. We have a systemic reswitch, if there are several alternative methods in possibly several industries, and a switch and a reswitch occur both on the envelope.

If the probability of isolated reswitching tends to zero, so will, it appears, the probability of systemic reswitching and of reverse capital deepening, for an isolated reswitch always underlies systemic reswitching and, by the way, also quite obviously multiple switching with more than two switch-points of two wage curves. It further underlies reverse capital deepening, for this means that the wage curves corresponding to two methods for producing commodity 1, all methods in the other industries being equal, intersect twice, and the second switch (at the higher rate of profit) is on the envelope, while the first switch is dominated by at least one other wage curve. If the probability of isolated reswitching tends to zero in large systems, reverse capital deepening at a given switch will do the same, for if it did not, one could in each case consider the underlying isolated reswitches by assuming away the technique with the wage curve dominating the first switch and would thus obtain more cases of isolated reswitching, so that the probability of isolated reswitching would not approach zero.

In other words: The probability for a systemic reswitch is much lower than that for an isolated reswitch. Every systemic reswitch is also an isolated reswitch, but not conversely, because at least one of two switch-points on the envelope of an isolated reswitch is likely to get dominated by some other wage curve, as soon as we have a multiplicity instead of only two techniques and wage curves. We shall see in the last section that the probability of reverse capital deepening is in between the probabilities for isolated and systemic reswitching. More complicated results obtain, if one looks not at one given switch, but at all switch-points of a system with many techniques taken together. Will always at least one case of reverse capital deepening appear among the possibly many switch-points of a large system? This question will be addressed at the end of the paper. For wider issues, in particular for an account of how my views have changed in consequence of successive new analytical findings, I must refer to other papers.[1]

[1] A referee, to whom I owe thanks, suggested that I amend this paper with an overview of how the analytical discoveries were made and of how I interpreted them, but that might lead far away from the present argument and confuse the issue. Moreover, such an overview regarding the prehistory of this paper has been written already and is in print (Schefold 2022a), while broader conclusions from the paper with Kersting and from the thesis here presented are proposed in Schefold (2021a). The referee also suggested that I say more about the difference between pure and applied theory and in particular about the problem of using Leontief matrices as empirical counterparts for Sraffa systems, but this has been discussed in my controversy with Fabio Petri, see Schefold (2022b) and Schefold (2022c). The referee thought that the title of this paper promised such explanations, but there is a misunderstanding here. "Explained" in the title does not refer to an intention to make the paper especially understandable to non-mathematical readers or to readers not acquainted with the previous discussions on capital theory. This paper is, and must be, a piece in mathematical economics. It provides the proofs of a number of theorems, which demonstrate the rarity of reswitching, each under special assumptions. "Explained" then expresses the claim that these theorems, taken together, explain the phenomenon of the rarity of reswitching. I say "explained" and not "proved" because it is a matter of judgement whether the assumptions cover a sufficiently broad number of cases. I try to argue that the assumptions are indeed sufficiently broad, they

2 The Behaviour of Prices and the Wage Rate

We shall have to use properties of Sraffa systems which ensure the quasi-linearity of prices and of wage rates as functions of the rate of profit. Much research has been dedicated to the task of specifying properties of Sraffa systems, which lead to only moderate movements of prices and relative prices in response to changes of distribution. These considerations can be traced back to Ricardo and his followers, ever since Ricardo realised in the preparation of his *Principles* that a rise of the wage rate and, with given methods of production, a consequent fall of the rate of profit would lead to a rise of the prices of labour-intensive industries relative to that of capital-intensive industries so that, if a suitable average price remained constant, prices of the capital-intensive industries would fall against ordinary intuition.

Recent decades have seen a rise in the number of papers devoted to empirically estimating the response of prices to changes of the rate of profit. Some of the main findings are to be summarised here. In a seminal paper, Bienenfeld (1988) estimates Sraffian standard prices from 71-sectoral US input-output tables of eight different years and compares these prices, termed 'exact', to theoretically derived linear and quadratic approximations. When computing the mean absolute deviations (MADs) of the approximated prices from the exact ones at different points over the range of the profit rate of each price curve, he finds the MADs of the linear approximation to vary between only 0.05% and 17.20%, indicating remarkably low deviations of prices from linear prices. The quadratic approximation fits the actual prices even better, yielding MADs between 0.01% and 0.27% (Bienenfeld, 1988, Table 1). The stunning accuracy of Bienenfeld's approximation is confirmed by Iliadi et al. (2014) with more recent input-output data from Denmark, Finland, France, Germany, and Sweden. Another one of their empirical results is the rarity of non-monotonic movements of standard prices in response to changes of the profit rate; only 18.8% of the computed standard prices are non-monotonic functions of the rate of profit (Iliadi et al., 2014, p. 47).

A further aspect that has attracted the interest of scholars in the field is the deviation of empirical Sraffian production prices from labour values. Using Chinese input-output data of the year 1997, Mariolis and Tsoulfidis (2009) estimate the difference between Sraffian standard prices and labour values at the empirically observed rate of profit. Averaging over all 38 sectors, they compute the mean absolute deviation of standard prices from labour values to be 11.2% (Mariolis and Tsoulfidis, 2009, p. 12). Similar results are obtained by Tsoulfidis (2008) for the Japanese economy of five years between 1970 and 1990. At the actual rate of profit, the MAD of prices of production from labour values ranges between 8.9% and 11.7% for the five years (Tsoulfidis, 2008, p. 715). These results are in line with computations for the Greek (Tsoulfidis and Maniatis, 2002) and US (Ochoa, 1989; Shaikh, 1998, 2016) economies, often interpreted as empirical support for the labour theory of value, see, e.g., Shaikh (1998).

Turning to wage curves, empirical results based on input-output data from different

certainly are better than the old boundedness assumption and it is here that the empirical results come in, but readers can ignore the empirical section and judge the results on the basis of their theoretical views alone, if they wish. I should highly welcome an analysis of whether the hypotheses are reasonable and the arguments are sound – more reflection on their significance may follow afterwards.

countries and years have consistently shown only mild deviations of wage curves from linearity. Using German data of different years provided by Krelle (1977), Petrović (1991, p. 105) estimates the absolute value of the correlation between wage and profit rates to vary between 97.77% and 98.87%, indicating minor deviations from a linear relationship between wages and the profit rate. He repeats this computation with Yugoslav input-output data from 1976 and 1978 and pays special attention to the impact of different numéraires on the shape of wage curves. A major result is that wage curves tend to be even closer to straight lines when composite commodities are used as the numéraire (Petrović, 1991, p. 105). This result is significant, given the fact that net output is often used as the numéraire, and will, in realistic settings, be a composite commodity. Findings of almost linear wage curves have also been obtained for, among others, the Brazilian (da Silva, 1991), Italian (Marzi, 1994), Korean (Tsoulfidis and Rieu, 2006), and US (Ochoa, 1989; Shaikh, 2016) economies. Iliadi et al. (2014, fn. 1) provide a helpful overview of further related empirical literature.

These results call for an explanation, using earlier work (Schefold, 2022a). Let $\mathbf{A} \geq 0$ be a non-negative basic input matrix that is diagonalisable and $\mathbf{l} > 0$ the associated labour-vector so that the system $(\mathbf{A}, \mathbf{l})$ is regular in the sense of Schefold (1971). If prices

$$(1 + r)\mathbf{A}\mathbf{p} + w\mathbf{l} = \mathbf{p}$$

are expressed in terms of Sraffa's standard commodity, the wage curve is linear and $w = 1 - (r/R)$, where $R > 0$ is the maximum rate of profit and $1/(1 + R)$ is the dominant eigenvalue of $\mathbf{A}$. The eigenvalues are ordered according to modulus, $\mu_1 = 1/(1 + R) \geq |\mu_2| \geq \cdots \geq |\mu_n| > 0$. Using the right-hand side eigenvectors $\mathbf{x}^1, \ldots, \mathbf{x}^n$ and representing the labour vector as a linear combination of these eigenvectors, $\mathbf{l} = \alpha_1\mathbf{x}^1 + \cdots + \alpha_n\mathbf{x}^n$, we can write standard prices as

$$\mathbf{p} = \left(1 - \frac{r}{R}\right) (\mathbf{I} - (1 + r)\mathbf{A})^{-1} \mathbf{l} = \frac{R - r}{R} \sum_{i=1}^{n} \frac{\alpha_i}{1 - (1 + r)\mu_i} \mathbf{x}^i. \qquad (1)$$

If one supposes that the $|\mu_i|$ are small, $i = 2, \ldots, n$, the standard prices become, for each component, approximately a linear function of the rate of profit:

$$\mathbf{p} \cong \left(\frac{1 + R}{R}\right) \alpha_1\mathbf{x}^1 + \left(1 - \frac{r}{R}\right) (\alpha_2\mathbf{x}^2 + \cdots + \alpha_n\mathbf{x}^n) = \mathbf{a} + r\mathbf{b}, \qquad (2)$$

where $\mathbf{a} = \mathbf{l} + \frac{\alpha_1}{R}\mathbf{x}^1$, $\mathbf{b} = -\frac{1}{R}(\mathbf{l} - \alpha_1\mathbf{x}^1)$. Prices are constant, if $\mathbf{b} = 0$, which means that the labour vector is the Frobenius right-hand side eigenvector of the matrix – then, the labour theory of value holds. If this is not the case and if none of the $\alpha_i\mathbf{x}^i$ is negligibly small, there are two ways to rationalise the procedure. On the one hand, the empirical investigations have shown that standard prices do not deviate much from linearity. The dominant eigenvalue is not small. Formula (1) therefore demonstrates that standard prices can be near-linear, with all $\alpha_i\mathbf{x}^i$ being of significant size, *only* if the non-dominant eigenvalues are small. On the other hand, this condition is also *sufficient*, and the question arises, under which circumstances this condition will be fulfilled. We know from the Goldberg-Neumann theorem (Goldberg and Neumann, 2003) that the eigenvalues will be small, if

A is a random matrix (Schefold, 2013a). The randomness of **A** therefore is sufficient for the quasi-linearity of standard prices. In a sense, it is also necessary. For if standard prices are strictly linear, the non-dominant eigenvalues must vanish completely and **A** is of rank one. Every semi-positive matrix of rank one must be positive, as is easy to see, and can therefore be written as $\mathbf{A} = \mathbf{cf}$, where $\mathbf{c} > 0$ is a column vector and $\mathbf{f} > 0$ a row vector. Now it is important to understand that our matrix **A** with small eigenvalues does not have to be close to matrix **cf** in the sense that each element a_{ij} of **A** would have to be close to the corresponding element of **cf**, that is to $c_i f_j$. It is not necessary that $|\mathbf{A} - \mathbf{cf}| < \epsilon$ for some $\epsilon > 0$. The conditions of the Goldberg-Neumann theorem (Goldberg and Neumann, 2003) imply that the rows of **A** must have coefficients with a distribution that is i.i.d. with mean c_i, as is shown in the Appendix to Schefold (2013a). Because of this distribution, matrix **A** is even simpler and is "close" to **ce**, where $\mathbf{e} = (1, \ldots, 1)$, but this must hold only on average. The individual rows $\mathbf{a}_i$ of **A** can have elements that are very different from c_i, provided only that they are non-negative and their mean is equal to c_i. In particular, many elements of $\mathbf{a}_i$ can be equal to zero, provided only that the mean equals c_i. The presumption that actual input-output systems have a random structure becomes utterly implausible, if one does not understand that **A** must be "close" to **ce** only on average. Further, it should be kept in mind that the conditions of the Goldberg-Neumann theorem are sufficient conditions for the non-dominant eigenvalues to be small for large systems; they are not necessary, as is clear from the fact that $\mathbf{A} = \mathbf{cf}$ also has the property that all non-dominant eigenvalues vanish, and this property presumably still holds, if we perturb the coefficients of **cf**, but the extent to which that would be possible is not known, if $\mathbf{f} \neq \mathbf{e}$.

To summarise: the near-linearity of standard prices results, if the non-dominant eigenvalues are small, and this condition is also, in essence, necessary (setting apart certain special cases such as that of the labour theory of value). The randomness of the input matrix is sufficient for this result, and I have argued that it is also, in essence – but not strictly – necessary. The systems under consideration must be large.

It should be noted that the *property* of standard prices being near-linear and not constant depends only on the eigenvalues, not on the labour vector. However, the *direction* into which near-linear standard prices move depends a great deal on the labour vector, as is obvious from formula (1) and the representation of **l** by the eigenvectors. For instance, it is possible that some of the α_i are close to or equal to zero. If $\alpha_2 = \cdots = \alpha_n = 0$, we have the case of the labour theory of value and prices are a constant function of the rate of profit and this will then hold even if all non-dominant eigenvalues are not small. The insight suggests a generalisation. We consider the k-th component p_k of prices **p** in (1).

Proposition

Using the notation and the assumptions (in particular: **A** *is diagonalisable), price* p_k *is linear (close to linearity), if and only if* $\mu_i \alpha_i = 0$ *(*$\mu_i \alpha_i$ *is small);* $i = 2, \ldots, n$*; where we assume that the* $\mathbf{x}^i$ *are so normalised that* $x_k^i = 1$.

The proof is obvious. The Proposition is important, because it demonstrates that prices can be linear, even if the system is not random. If, as appears empirically to be the case, the first non-dominant eigenvalues are not small, prices can be linear all the same, if the corresponding α_i are small. The α_i add up to l_k. Hence the individual α_i are likely to be small, if n is large. So it becomes plausible that price curves are quasi-linear even if some non-dominant eigenvalues are not small.

If standard prices are linear functions of the rate of profit, relative prices and prices in other standards become hyperbolas, and the extent to which they deviate from linearity for $0 \leq r \leq R$ requires a special investigation.

A curious role is played by prices in terms of the wage rate $\hat{\mathbf{p}} = \mathbf{p}/w$. If we divide (2) by $w = 1 - (r/R)$, the components of the price vector are *algebraically* hyperbolas, but the trajectory described by the vector $\hat{\mathbf{p}}$ in space is, *geometrically*, a straight line in the space of prices.

Let a numéraire vector be given that is, in general, not the standard commodity. With the normalisation $1 = \mathbf{dp} = \mathbf{d}\hat{\mathbf{p}}w$, we obtain for the wage rate in this standard a somewhat complicated expression. It can be simplified, using the representation of $\mathbf{d}$ by the left-hand eigenvectors of $\mathbf{A}$, $\mathbf{d} = \beta_1\mathbf{q}_1 + \cdots + \beta_n\mathbf{q}_n$. We add the important assumption that none of the α_i and none of the β_i is equal to zero. *Then* we can use the strong normalisation (Schefold, 2013a) and assume that the scales of the eigenvectors $\mathbf{q}_i$ and $\mathbf{x}_i$ are chosen in such a way that $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 1$. Using the fact that left-hand and right-hand eigenvectors belonging to different eigenvalues are orthogonal, one obtains for the wage curve w

$$
\begin{aligned}
w = \frac{1}{\mathbf{d}\hat{\mathbf{p}}} &= \frac{1}{\sum_{i=1}^{n} \dfrac{\mathbf{dx}^i}{1-(1+r)\mu_i}} = \frac{R-r}{(1+R)\mathbf{q}_1\mathbf{x}^1 + (R-r)(\mathbf{q}_2\mathbf{x}^2 + \cdots + \mathbf{q}_n\mathbf{x}^n)} \\
&= \frac{R-r}{(1+R)\mathbf{q}_1\mathbf{x}^1 + (R-r)\mathbf{mv}},
\end{aligned}
\tag{3}
$$

where we have introduced the vectors $\mathbf{m}$ and $\mathbf{v}$ for the deviation between the numéraire and the left-hand side Frobenius eigenvector and the deviation between the labour vector and the right-hand side Frobenius eigenvector, in formulas $\mathbf{m} = \mathbf{d} - \mathbf{q}_1 = \mathbf{q}_2 + \cdots + \mathbf{q}_n$, $\mathbf{v} = \mathbf{l} - \mathbf{x}^1 = \mathbf{x}^2 + \cdots + \mathbf{x}^n$. These vectors are expressions for the deviation of the numéraire from the standard vector (in the strong normalisation) that would yield a linear wage curve, if it was taken as the numéraire, and the deviation of the labour-vector from that eigenvector (in the strong normalisation) that would, if it were the labour vector, also result in a linear wage curve. It is clear from (3) that the wage curve will be linear *if and only if* $\mathbf{mv} = 0$. Sufficient conditions for linearity would be $\mathbf{m} = 0$ or $\mathbf{v} = 0$, and these are, in our notations, conditions for using a vector proportional to the standard commodity as numéraire – and then the wage curve is linear – or assuming a uniform composition of capital, which, as is well known, will be the case if and only if the labour vector happens to be the right-hand side eigenvector of the matrix.

But these conditions are not necessary. We arrived at standard linear prices by assuming that the matrix was random. We now make a similar assumption that turn the components of the numéraire vector and of the labour vector into random variables. More precisely, the components of $\mathbf{m}$ and the components of $\mathbf{v}$ shall be random variables, written as vectors for convenience, but they are not random vectors. We assume that the deviations of the numéraire vector from the standard and of the labour vector from the labour vector yielding the uniform composition of capital are uncorrelated, since the technical necessities represented by the labour vector and the choice of the numéraire relative to the standard vector can be thought to be independent, and this independence should show in large systems. Hence we assume that $\text{cor}(\mathbf{m}, \mathbf{v}) = 0$, hence that the covariance $\text{cov}(\mathbf{m}, \mathbf{v}) = 0$. If this is the case, one has the well known consequence $\mathbf{mv} = n\bar{m}\bar{v}$, where $\bar{m}$ and $\bar{v}$ are the arithmetic means of the random variables consisting of the components of $\mathbf{m}$ and $\mathbf{v}$.

It follows that, if the covariance condition is fulfilled, the wage curve will be linear *if and only if* $\bar{m}$ or $\bar{v}$ or both are zero. Now it turns out that, if the system is random, $\bar{v}$ will be zero. For we can write, using Schefold (2016, p. 15),

$$n\bar{v} = \mathbf{e}(\mathbf{l} - \mathbf{x}^1) = \mathbf{ex}^2 + \cdots + \mathbf{ex}^n.$$

Since $\mathbf{A}$ is random, the left-hand eigenvector of $\mathbf{A}$ will be proportional to $\mathbf{e}$ for large n, since the components on the rows of $\mathbf{A}$ are independently and identically distributed. If $\mathbf{e}$ is the left-hand eigenvector of $\mathbf{A}$, it is orthogonal to $\mathbf{x}^2, \ldots, \mathbf{x}^n$ and $n\bar{v} = 0$. Hence $n\bar{m}\bar{v} = 0$ and $\mathbf{mv} = 0$. Note the converse result, which follows: If the system is random and the wage curve is linear, we have $\bar{v} = 0$ and $\mathbf{mv} = 0$, hence $mv - n\bar{m}\bar{v} = \text{cov}(\mathbf{m}, \mathbf{v}) = 0$. This means that, if one accepts that $\mathbf{A}$ is random and that wage curves are linear, one must also accept the covariance condition – a mathematical result to be noted by the critics of the covariance-condition.

The crucial condition that has here been added to the randomness of $\mathbf{A}$ is that the covariance of $\mathbf{m}$ and $\mathbf{v}$ vanishes. This may be rewritten as $\text{cov}(\mathbf{m}, \mathbf{v}) = \text{cov}(\mathbf{d}, \mathbf{l}) - \text{cov}(\mathbf{d}, \mathbf{x}^1) - \text{cov}(\mathbf{q}_1, \mathbf{l}) + \text{cov}(\mathbf{q}_1, \mathbf{x}^1)$. A significant correlation might here be expected for the last term in case the matrix were nearly symmetric. But if $\mathbf{A}$ is random and we represent $\mathbf{A}$ by its deterministic counterpart, the matrix is written as $\mathbf{ce}$, and $\mathbf{q}_1$ is proportional to $\mathbf{e}$ and $\mathbf{x}_1$ is proportional to $\mathbf{c}$. So, $\mathbf{e}$ and $\mathbf{c}$ may be considered as independent, and the covariance condition may be assumed and may, together with the randomness of $\mathbf{A}$, be regarded as the most plausible explanation proposed so far for the quasi-linearity of wage curves often, but not always, encountered in empirical investigations. As we have seen, each matrix $\mathbf{A} = \mathbf{ce}$ stands as deterministic counterpart for a large class of random matrices, namely those matrices where each row $\mathbf{a}_i \geq 0$ has coefficients with a distribution that is i.i.d. and a mean c_i. It remains to derive the condition for the labour vector, which is necessary and sufficient for a linear wage curve, given $\mathbf{A} = \mathbf{ce}$. Using $(\mathbf{I} - \mathbf{ce})(\mathbf{I} + \frac{\mathbf{ce}}{1-\mathbf{ec}}) = \mathbf{I}$ and $\mathbf{ec} = 1/(1+R)$, one gets

$$\mathbf{p} = w\left(\mathbf{I} - (1+r)\mathbf{ce}\right)^{-1}\mathbf{l} = w\left(\mathbf{I} + \frac{1+r}{(R-r)\mathbf{ec}}\mathbf{ce}\right)\mathbf{l},$$

$$w = \frac{R-r}{R\mathbf{dl} + (1+R)\mathbf{dcel} + r[(1+R)\mathbf{dcel} - \mathbf{dl}]}.$$

Hence, the wage curve is linear, if and only if

$$(\mathbf{dce} - \mathbf{ecd})\mathbf{l} = 0, \text{and if and only if } \mathbf{d}(\mathbf{cel} - \mathbf{lec}) = 0.$$

The labour theory of value results, if $\mathbf{l}$ is proportional to $\mathbf{c}$. The wage curve then is linear, as it is, if $\mathbf{d}$ is proportional to $\mathbf{e}$ (Sraffa's case), but much less is necessary. A linear wage curve also results, if $\mathbf{l}$ is orthogonal to the vector $\mathbf{dce}-\mathbf{ecd}$ or $\mathbf{d}$ is orthogonal to $\mathbf{cel}-\mathbf{lec}$. The wide range that is opened up if these conditions are fulfilled approximately explains the empirical finding that wage rates are quasi-linear. To relate these conditions to the covariance condition is more complicated and left for another occasion.

But it is worthwhile to analyse the linearity of the wage curve (3) also without assuming the randomness of $\mathbf{A}$. (3) can be rewritten as, with the normalisations $\mathbf{l} = \alpha_1\mathbf{x}^1 + \cdots + \alpha_n\mathbf{x}^n$, $\mathbf{d} = \beta_1\mathbf{q}^1 + \cdots + \beta_n\mathbf{q}^n$, $\mathbf{q}_i\mathbf{x}^i = 1$; $i = 2, \ldots, n$;

$$\frac{1}{w} - \frac{\alpha_1\beta_1}{1-(1+r)\mu_1} = \frac{\alpha_2\beta_2}{1-(1+r)\mu_2} + \cdots + \frac{\alpha_n\beta_n}{1-(1+r)\mu_n} = D(r)$$

The right-hand side of this equation is a residual, called $D(r)$. The wage curve is linear, if and only if $D = 0$. For this it is sufficient that $\alpha_2 = \cdots = \alpha_n$ (standard commodity case) or $\beta_2 = \cdots = \beta_n = 0$ (labour theory of value case) or that the products $\alpha_i\beta_i$ disappear: if α_i is not zero, β_i should vanish and conversely; this is a mixed case, Moreover, if we are dealing with approximations ($\alpha_i\beta_i$ small, but not zero), it helps, if μ_i is small, for then $1-(1+r)\mu_i$ will not become small, leading to a large $D(r)$, as r increases from zero to R. Hence we can formulate:

Proposition
The wage curve will be quasi-linear, if $\alpha_i\beta_i\mu_i$ remains small; $i = 2, \ldots, n$.

This criterion is useful in empirical applications; α_i, β_i, μ_i are then indicators, which characterise systems with linear wage curves.[2] However, we cannot really say that small α_i, β_i, μ_i; $i = 2, \ldots, n$; are "causes" of quasi-linear wage curves, since a small $\alpha_i\beta_i\mu_i$ has no immediate economic meaning, whereas the randomness of $\mathbf{A}$ and the covariance-condition have an interpretation: the specific structure of production in a single industry appears as accidental in a large system.

The quasi-linearity of prices and wage rates, which we have here tried to explain, stands in contrast to Sraffa's (1960) results, who emphasizes the variability of relative prices, in particular of non-basics, and speaks in his chapter on changes of technique of a "rapid succession of switches" (Sraffa, 1960, p. 85), as one moves down the envelope, resulting from a large number of alternative techniques with wage curves that are far from linear. I have indicated in the first chapter of this paper how, following Kersting and Schefold (2021), one must qualify this affirmation. There is only a small number of efficient techniques in the relevant range of the rate of profit according to theoretical considerations, empirical results and numerical experiments.

A similar discrepancy arises with respect to the variability of prices. Sraffa derives his very elegant formula for the reduction to dated quantities of labour, with prices expressed

[2]The idea of considering the products $\alpha_i\beta_i\mu_i$ as indicators is taken from Ferrer-Hernández and Torres González (2022). They use vertically integrated systems with maximum rates of profit normalised to $R = 1$. This has here been avoided, since the analysis of reswitching requires to start from the systems themselves, and the comparison of systems on envelopes of wage curves is not possible with normalised maximum rates of profit.

in terms of the standard commodity:

$$\mathbf{p} = \sum_{t=0}^{\infty} w(1+r)^t \mathbf{L}^t = \sum_{t=0}^{\infty} \left(1 - \frac{r}{R}\right)(1+r)^t \mathbf{A}^t \mathbf{l}. \tag{4}$$

Sraffa argues that the indirect quantities of labour (the labour expended t periods ago and indirectly embodied in the present product) $\mathbf{L}^t = \mathbf{A}^t\mathbf{l}$ may be quite small, if one refers to distant periods, but that they may make their influence felt near the maximum rate of profit, where cumulative profits on the corresponding advance of wage costs, expressed by $(1+r)^t$, is large. Their weight depends on the 'terms' $(1-r/R)(1+r)^t$, which exhibit sharp maxima for large t. These maxima have to be multiplied by the corresponding labour input, which is small, but the product of the maximum and the labour input may be large, Sraffa suggests, for he draws a series of such terms, with the labour inputs chosen so that subsequent maxima seem to be of equal height. This impresses the reader, for if different capital goods have indirect labour inputs, that are unequally distributed over the past, their relative value must fluctuate unpredictably, as the rate of profit is varied.

However, capital goods are essentially basics, and only for non-basics or in Austrian models is it possible to conceive of the past labour inputs to be distributed erratically over time. If capital goods are basic, the sequence of past labour inputs will diminish with certain regularities, as t increases. Moreover, a graphic representation of the 'terms' will then look quite different from the figure presented by Sraffa; the maxima will be below the linear wage curve. A diagram for the 'terms' for commodities that are basic has been drawn in Schefold (2021b) and one can use an approximation for standard prices of basic commodities, which shows that they are, apart from the first few periods, say T, characterised by a regular parallel reduction. As is well known, the powers of $\mathbf{A}$, under the assumptions we have made for this matrix, $\mathbf{A}^t$, diminish rapidly, but we get convergence, if we multiply by the corresponding power of $1+R$; we have $(1+R)^t\mathbf{A}^t \xrightarrow[t\to\infty]{} \bar{\mathbf{p}}\bar{\mathbf{q}}$, where $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are normalised right-hand and left-hand Frobenius eigenvectors of $\mathbf{A}$ with $\bar{\mathbf{q}}\bar{\mathbf{p}} = 1$. One thus obtains the following modified formula for the reduction to dated quantities of labour:

$$\begin{aligned}\mathbf{p} &= \sum_{t=0}^{\infty} \left(1 - \frac{r}{R}\right)(1+r)^t \mathbf{A}^t \mathbf{l} \\ &= \left(1 - \frac{r}{R}\right)\sum_{t=0}^{T}(1+r)^t \mathbf{A}^t \mathbf{l} + \left(1 - \frac{r}{R}\right)\sum_{t=T+1}^{\infty}\left(\frac{1+r}{1+R}\right)^t \bar{\mathbf{p}}\bar{\mathbf{q}}\mathbf{l} + \mathbf{z}.\end{aligned} \tag{5}$$

The vector $\mathbf{z}$ is a vector of residuals, which tend to zero, if T is sufficiently large. For details see Schefold (2021b).

The reader can verify, taking the different normalisations into account, that (2) and (5) coincide for $T = 0$. This relationship, which has not been noted before, is remarkable, not only because (2) and (5) look very dissimilar at first sight. They also belong to different analytical approaches. (2) is based on the assumption that the non-dominant eigenvalues are small, a plausible explanation of this condition being that the matrix is random. (5) is based on the older tradition of expressing functions by means of an infinite (here geometrical) series, with the only difference that, instead of omitting higher terms for an approximation, higher terms are approximated using the convergence of $(1+R)^t\mathbf{A}^t$ to $\bar{\mathbf{p}}\bar{\mathbf{q}}$. Obviously, with $T = 0$, the latter approximation plays a decisive role. Nonetheless, (5) seems to be independent of randomness assumptions; the formula only presupposes that $\mathbf{A}$ is semi-positive, indecomposable, productive and diagonalisable – in short: $\mathbf{A}$ is essentially an ordinary Sraffa matrix. How can (2) and (5) then become equivalent?

The immediate answer, of course, is that (2) and (5) with $T = 0$ can be bad approximations, if the non-dominant eigenvalues are not small. However, there is another argument. There is hidden randomness in (5), even if T is large, and not only in (5). It is also in Sraffa's own equation (4), insofar as the powers $\mathbf{A}^t$ are matrices for which the non-dominant eigenvalues – we denote them by $\mu_{t,2}, \ldots, \mu_{t,n}$ – tend to zero relative to the dominant eigenvalue of $\mathbf{A}^t$, denoted by $\mu_{t,1}$. For we have, if the eigenvalues of $\mathbf{A}$ are $\mu_1, \ldots, \mu_n; \mu_1 > |\mu_2| \geq \cdots \geq |\mu_n| > 0$; and if $\mathbf{A}$ is primitive so that $\mu_1 > |\mu_2|$, $\mu_{t,i} = (\mu_i)^t$, hence $|\mu_{t,i}/\mu_{t,1}| \to 0; i = 2, \ldots, n$, for $t \to \infty$. So, the main characteristic of randomness, non-dominant eigenvalues tending to zero, sneaks in even in the case of ordinary primitive Sraffa matrices. The higher powers of $\mathbf{A}$ stand for repeated interactions between the prices of basics or the quantities, if one interprets the formula for activity levels $\mathbf{q} = \mathbf{d}(\mathbf{I} - \mathbf{A})^{-1} \cong \mathbf{d}(\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^t)$ as an iterative planning process in T steps. Specific information disappears in large systems that are not imprimitive.

Equation (2), which has now been buttressed with equation (5), assuming $T = 0$, provides an information about the behaviour of relative prices, which we shall need later. The rate of change of a price p_i, $\dot{p}_i/p_i$ – where $\dot{p}_i$ denotes the derivative – will become relevant for assessing whether the expression, which will play an important role in Sections 3-5,

$$z_i(r) = \frac{p_1(r)}{(1+r)p_i(r)}, i = 2, \ldots, n,$$

falls with r increasing from 0 to R. Assuming that the strong normalisation is possible and using it, we now express (2) or (5) for standard prices in linear approximation as

$$\mathbf{p} = \mathbf{l} + \frac{1}{R}\mathbf{x}^1 + \frac{r}{R}(\mathbf{x}^1 - \mathbf{l}).$$

On the one hand, we then get for the rate of change of an individual price

$$\frac{\dot{p}_i}{p_i} = \frac{x_i^1 - l_i}{Rl_i + x_i^1 + r(x_i^1 - l_i)},$$

so that $\dot{p}_i/p_i < 0$, if and only if $x_i^1 < l_i$; we can call a sector with this property labour-rich. On the other hand, we get a condition for $z_i(r)$ to fall in terms of the rates of change. Obviously, $\mathrm{d}z_i/\mathrm{d}r < 0$, if and only if

$$\dot{p}_1(1+r)p_i < p_1p_i + (1+r)p_1\dot{p}_i$$

or

$$\frac{\dot{p}_1}{p_1} < \frac{\dot{p}_i}{p_i} + \frac{1}{1+r},$$

where $\dot{p}_i/p_i < 0$, if and only if sector i is labour-rich, which sounds plausible. We distinguish four cases in the following Table 1.

		$\dot{p}_1$	
		> 0	< 0
$\dot{p}_i$	> 0	a	b
	< 0	c	d

Table 1: Conditions for $z_i(r)$ to fall.

If sector 1 and i in Table 1 are both labour-poor (case a), both prices rise. Insofar, there is an equal chance for z_i to rise or to fall, because $\mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p} = 1$ and therefore $\mathbf{q}(\mathbf{I} - \mathbf{A})\dot{\mathbf{p}} = 0$. Hence, because of the factor $1 + r$ in the denominator, $z_i(r)$ will fall in more than half of the cases, and the analogous conclusion holds for (d). It is certain that $\mathrm{d}z_i/\mathrm{d}r < 0$ in case (b). It would be certain that $z_i(r)$ rises, if the factor $1 + r$ was absent; but since it is not, one will get a falling $z_i(r)$ in some cases even in case (c), that is, if sector 1 is labour-poor, but less so than sector i. In conclusion, $z_i(r)$ will fall more often than it rises, on average, considering many large systems. This will be important in subsequent sections. The conclusion is based on a probabilistic consideration, which one would perhaps not like to introduce in pure economic theory, but, in modern economics, probability arguments are used very often.

To make inferences from trends based on averages, possibly disturbed by events, is the approach of the modern applied economist. Sraffa wanted to alert the community of economists to the fact that the *pure* neoclassical theory was flawed and that the factor 'capital' could, unlike land and labour, not be measured prior to the determination of prices. It was not the only problem of pure neoclassical theory, but, in the form of reswitching, the one that induced the most profound, at any rate effective critique. In the meantime, the mainstream has changed, the critique must adapt to the fact that the standards of rigour are different for applied economics. The roots of an overestimation of the reswitching and related arguments are in Sraffa's book itself and due to his strategy which addressed an audience, which was different from that with which we are confronted today.

There seems not to be much room for reswitching, if all wage curves are really close to linearity, but we have argued that a sufficient number of them are sufficiently close to linearity to warrant our conclusions about $z_i(r)$. Near linearity of wage curves is not sufficient to rule out reswitching, however. Numerical examples show that the curves intersecting at least twice are often quasi-linear and close together (Mainwaring and Steedman, 2000). This explains why we cannot stop here but need a deeper analysis. The following investigation is concerned with showing that the rarity of reswitching – an empirical phenomenon, both in the isolated form and as associated with reverse capital deepening – is in part to be explained by factors connected with randomness, but also by other influences like the geometric properties of the reswitching body.

3 The Reswitching Body

We first must elaborate the formal apparatus already introduced in more detail. We assume that a productive and indecomposable system is given, represented by a semi-positive input-output matrix of order n $\mathbf{A} \geq 0$ and a positive labour vector $\mathbf{l} > 0$, with processes $(\mathbf{a}_i, l_i)$; $i = 1, \ldots, n$; as the methods of production of the n industries. As is well known, the normal prices of this system $\mathbf{p}$ will be positive for all rates of profit r between 0 and a maximum rate of profit R, and the wage rate w will fall monotonically from a maximum at $r = 0$ to zero at $r = R$, where prices are expressed in some commodity standard $\mathbf{d}$ with $\mathbf{dp} = 1$. We recall that Sraffa's standard prices are defined by the condition $\mathbf{d} = \mathbf{q}(\mathbf{I} - \mathbf{A})$, where $\mathbf{q}$ is normalised by subsequently putting $\mathbf{el} = 1$; $\mathbf{e} = (1, \ldots, 1)$; and $\mathbf{ql} = 1$ so that the normal prices

$$\mathbf{p} = (1 + r)\mathbf{Ap} + w\mathbf{l} \tag{6}$$

result from

$$\mathbf{p} = w(\mathbf{I} - (1 + r)\mathbf{A})^{-1}\mathbf{l}$$

and the wage rate is $w = 1 - \frac{r}{R}$, if prices are expressed in terms of the standard commodity, for we then have

$$1 = \mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p} = r\mathbf{qAp} + w\mathbf{l} = \frac{r}{R}\mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p} + w\mathbf{ql} = \frac{r}{R} + w.$$

We shall also often need prices in terms of labour commanded or in terms of the wage rate $\hat{\mathbf{p}} = \frac{\mathbf{p}}{w} = \frac{R}{R - r}\mathbf{p}$.

We assume a stationary state, in which a vector of a surplus $\mathbf{s} \geq 0$ is produced, given activity levels $\mathbf{q}$ such that $\mathbf{q}(\mathbf{I} - \mathbf{A}) = \mathbf{s}$ and $\mathbf{q} = \mathbf{s}(\mathbf{I} - \mathbf{A})^{-1} > 0$. In this stationary state, the capital-labour ratio can be read off from the wage curve, if one assumes that a third form of numéraire is taken; it is set equal to the surplus: $\mathbf{d} = \mathbf{s}$. With this assumption, output per head y, $y = \mathbf{dp}/\mathbf{ql}$, is the same for all rates of profit. At the same time, it equals the sum of wages per head and profits per head, hence $y = w + rk$, where k is the capital-labour ratio $k = K/L = \mathbf{qAp}/\mathbf{ql}$, so that the capital-labour ratio can be read off from the wage curve diagram $k = (y - w)/r$. The capital-intensity is constant along the wage curve, if and only if the curve is linear; k is then equal to the absolute value of the slope of that curve. If the wage curve is convex, the intensity of capital falls, as the rate of profit rises. This is a so-called neoclassical Wicksell-effect. In the opposite case, the intensity of capital rises with the rate of profit. The intensity of capital then increases, as the wage falls. This is a non-neoclassical or, sometimes, a 'perverse' Wicksell-effect.

If another method of production is available in one of the industries, its adoption leads to another wage curve. For instance, process $(\mathbf{a}_1, l_1)$ is replaced by process $(\mathbf{a}_0, l_0)$. In the comparison of wage curves w^0 and w^1, that technique has to be chosen at each rate of profit which yields the highest real wage, and it can be shown that surplus profits induce the change-over to the better technique. If the wage curves are linear, higher rates of profit and lower wage rates mean that successively techniques of lower capital-intensities and higher labour-intensities will be adopted, but if the wage curves are curved, two techniques may have more than one switch-point in common. If wage curves w^0 and w^1 have two switch-points in common at rates of profit r_1 and r_2, $0 \leq r_1 < r_2 \leq R_i$, $R_i = \min(R_1, R_2)$, we speak of reswitching.[3] We speak of reverse capital deepening, if the first of these

[3]Usually, one speaks of reswitching only, if both switch-points are on the envelope. We called this

two switch-points is dominated by a third wage curve, which is, however, inferior at the second switch-point. The intensity of capital then rises at the second switch-point. The cases of reverse capital deepening are therefore dependent on two wage curves crossing on the envelope such that there is a "hidden" intersection of these same two wage curves below the envelope at a lower rate of profit. Insofar, reswitching in the sense of a double switch of two wage curves, the techniques differing in the method of production in one industry only, is the basic phenomenon. We called this isolated reswitching in Section 1, if one abstracts from the other techniques and their wage curves, and we called it systemic reswitching, if both switch-points are on the envelope. Reverse capital deepening seems to occur more often in reality than systemic reswitching, since it is quite likely that the earlier switch-point will be dominated, if there is a sufficient number of techniques available in various industries, the wage curves of which reach the envelope.

We speak of capital reversals, if, quite generally, an increase in the rate of profit leads not to a fall, but to a rise of the intensity of capital. Capital reversals may be due either to non-neoclassical Wicksell-effects or to reverse capital deepening and reswitching. The latter two are regarded as more fundamental, since Wicksell-effects are numéraire-dependent: A mere change of numéraire can, for the same technique, turn the neoclassical Wicksell-effect into a non-neoclassical Wicksell-effect or vice versa. For this reason and because the at least twofold intersection of wage curves is at the root also of reverse capital deepening, we here concentrate on isolated reswitching.

The paradox in question is interesting as a macroeconomic phenomenon, but it may also have a sectoral aspect, which is less known. To see this, it is useful to formally extend the analysis to negative rates of profit, for then the same wage curves, only extended to the interval $-1 \leq r \leq 0$, can be used to analyse the sectoral aspect. No economic meaning needs to be associated with a negative r. This has been examined in detail in Han and Schefold (2006). Here it suffices to observe two wage curves, which differ because the methods of production differ in one of the industries, which intersect for some r, $-1 \leq r \leq 0$. The two wage curves shall extend without a crossing, each to its maximum rate of profit; both maximum rates shall be positive. Of two wage curves, which do not intersect at all, the higher one will be better, because it has a higher maximum rate of profit. It will also be better, insofar as the wage at $r = 0$ can be higher, therefore output per head is higher – this can be read off from the wage curve at $r = 0$. Finally, this better technique will use less direct labour in the sector, in which the two techniques differ in one method of production. This can be seen by looking at the price equations at $r = -1$. With $r = -1$, one obtains from equation (6) $\hat{\mathbf{p}}(-1) = \mathbf{l}$. The prices must be lower for the better technique, which is w^1, say. We have $w^1 > w^0$, because $\mathbf{q}^1\mathbf{l} < \mathbf{q}^0\mathbf{l}$ (numbering the

systemic reswitching in Section 1. The assumption that one switch is a crossing of wage curves on the envelope becomes important as soon as there are alternative methods of production available also in other industries. For it will then generically be the case that, if two wage curves cross on the envelope, only one method in one industry will change or switch at that rate of profit, while two intersections of wage curves below the envelope may belong to two systems that differ in the employment of several methods in several industries. The intuitive reason for this difference is simple: if we are on the envelope, the wage rate is maximal, given the rate of profit. If it changes, different methods will become profitable according to the changing state of distribution, and they will generically come up one by one in a piecemeal fashion. If the intersection is below/ between two systems using different methods in several industries, the optimal combination of these methods has still to be found.

activity level vectors in the same way as the wage curves). Since all other labour inputs are the same in both techniques except in the industry where the method change takes place, less labour is used in that industry.

If we now return to the example where w^1 and w^0 have one crossing between -1 and 0, we observe the curious effect that the technique, which is better at positive rates of profit, uses more labour in the sector, where the change of technique occurs. If we take the case of two techniques which have one switch-point between -1 and 0 and another between 0 and the maximum rate of profit, the technique which emerges after the second switch-point will be more labour-intensive in the aggregate and yet use less labour in the sector in which the change takes place. The higher labour-intensity then is entirely due to changes of relative prices. It occurs at the macroeconomic level, although less labour is used microeconomically in the sector concerned and, indeed, in the economy as a whole. But this effect is not numéraire-dependant, like ordinary Wicksell-effects. Although it is thus interesting to extend the study of switch-points to the interval $-1 \le r \le 0$, we focus on reswitching between zero and the maximum rate of profit in what follows. The sectoral effect was found to occur more often than reverse capital deepening in Han and Schefold (2006). It would deserve more attention.

As explained in the introductory sections, we estimate the probability of reswitching by measuring the set of potential techniques, which generate two or more switch-points, relative to the set of techniques which generate only one. The idea had been pursued already in Schefold (1976) with the result that reswitching turned out to be not a fluke, but to have a positive probability.

We begin with the set $M(r)$ of methods of production $(\mathbf{a}_0, l_0)$ that have a switch with the technique $(\mathbf{A}, \mathbf{l})$ by being exchangeable with method of production $(\mathbf{a}_1, l_1)$ in the first sector at the prices ruling at the rate of profit r. Hence this set $M(r)$ follows from equation (7):

$$M(r) = \{(\mathbf{a}_0, l_0) \ge 0 \,|\, (\mathbf{a}_0, l_0)\tilde{\mathbf{p}}(r) = \hat{p}_1(r)\}. \tag{7}$$

$\tilde{\mathbf{p}}(r)$ here is a column vector $\tilde{\mathbf{p}}(r) = ((1+r)\hat{p}_1(r), \ldots, (1+r)\hat{p}_n(r), 1)^T$. It was proved in Schefold (1976), and in a reduced form in Schefold (1971), that $\tilde{\mathbf{p}}(r)$ is a vector that assumes $n+1$ linearly independent values at $n+1$ different rates of profit $0 \le r_1 < \cdots < r_{n+1} < R$, and of course, $\tilde{\mathbf{p}}(r) > 0$, $0 \le r < R$. This is what I now call the fundamental neo-Ricardian theorem. It holds for regular systems. This means essentially that the labour theory of value does not hold in that $\mathbf{l}$ is not an eigenvector of $\mathbf{A}$. $M(r)$ is obtained as the intersection of an n-dimensional hyperplane orthogonal to $\tilde{\mathbf{p}}(r)$ with $\mathbb{R}_+^{n+1}$. Given r, the vertices of $M(r)$ can readily be calculated by putting $(\mathbf{a}_0, l_0) = z_i(r)\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{R}_+^{n+1}$. Inserting the $z_i(r)\mathbf{e}_i$ into (7); $i = 1, \ldots, n+1$; one obtains:

$$z_1(r) = \frac{1}{1+r}, z_i(r) = \frac{\hat{p}_1(r)}{(1+r)\hat{p}_i(r)}; i = 2, \ldots, n; z_{n+1}(r) = \hat{p}_1(r). \tag{8}$$

We here encounter the $z_i(r)$ discussed in Section 2, but we do not yet use the probabilistic argument that the $z_i(r)$ are likely to fall. We below show the same by means of geometry on the basis of the Lemma (statements ix and x). It follows geometrically from $\tilde{\mathbf{p}}(r) > 0$ that $z_1(r) > 0, \ldots, z_{n+1}(r) > 0$, $0 \le r \le R$. It follows from the fundamental neo-Ricardian theorem that the n-dimensional simplex $M(r)$ turns in space in such a manner that it is never contained in any n-dimensional subspace for any interval, in which r moves.

Now we can express the possibility of reswitching. It occurs for all potential techniques in the intersection of $M(r_1) \cap M(r_2)$; $0 \leq r_1 < r_2 \leq R$. This intersection is an $(n-1)$-dimensional hyperplane, restricted to the $\mathbb{R}_+^{n+1}$, as the intersection of two n-dimensional simplices. A question then is how probable it is that we find an actual technique, that is in $M(r_1)$ and also in $M(r_2)$, if r_1 and r_2 are given. The answer is: It is improbable, since the n-dimensional measure $\mu(M(r_1) \cap M(r_2))$ is zero, and zero in particular relative to the n-dimensional measure of $\mu(M(r_1))$, which is positive.

But this does not mean that the probability of reswitching generally is zero, since, given r_1, we can ask whether reswitching will turn up if we *vary* r_2. It was shown in Schefold (1976) that the measure of all the possibilities then is positive. We define

$$M^\star = \bigcup_{\substack{0 \leq r_2 \leq R \\ r_2 \neq r_1}} M(r_1) \cap M(r_2).$$

This is the reswitching body. Its measure will be positive, $\mu(M^\star) > 0$, if the intersection of $M(r_2)$ with $M(r_1)$ covers an open n-dimensional neighbourhood on $M(r_1)$.

Actually, we could iterate this procedure. There could be more than two switch-points. We should find the set of points generating m switch-points, $m < n$, by considering the intersection

$$M_m^\star = M(r_1) \cap \cdots \cap M(r_m); 0 < r_2 < \cdots < r_m < R; r_1 \neq r_i \, \forall i.$$

Again, this set is of measure zero relative to $M(r_1)$, if the $r_2, \ldots, r_m$ are given, but if they are variable and if the $M(r_i)$ twist in space, they will in the end cover an open neighbourhood in $M(r_1)$, and obviously these sets must be contained in each other with $M(r_1) \supseteq M^\star \supseteq M_3^\star \supseteq M_4^\star \supseteq \cdots \supseteq M_m^\star$. The sets will be nested like Russian dolls.

It is clear that one point will be in common to all these sets, it is $(\mathbf{a}_1, l_1)$. For $M_m^\star$ is given by equations (7), taken for m different rates of profit. Because of the fundamental neo-Ricardian theorem, these equations will all be independent and the solutions will be an $(n+1-m)$-dimensional set, restricted to non-negative values. Solutions for higher m will be contained in those for lower m. If m rises to $n+1$, the solutions of the corresponding equations (7) will consist of one point only, because we will have as many unknowns as we have independent equations, and this solution can only be $(\mathbf{a}_1, l_1)$, which belongs to all these sets.

This plethora of switch-points may appeal to the critics of capital theory, but we shall see that the possibilities even for only reswitching shrink as the number of sectors increases. Unfortunately, to determine the measure of the reswitching body is not an elegant mathematical problem. Or it is my fault that I have not been able to render it more tractable. I apologise for the intricacies that follow.

Lemma. *Properties of the reswitching body.*

(i) If $(\mathbf{a}_1, l_1) > 0$*,* $M(r_2)$ *bisects* $M(r_1)$ *into two parts, each not empty, seperated by a hyperplane of dimension* $n-1$*, containing* $M(r_1) \cap M(r_2)$*, which is convex.*

(ii) *$M^\star$ does not cover the simplex $M(r_1)$: $M(r_1) - M^\star \neq \emptyset$.*

(iii) *$M^\star$ is star-shaped, in that every point is connected with $(\mathbf{a}_1, l_1)$: $P \in M^\star, P_1 = (\mathbf{a}_1, l_1) \Rightarrow \overline{PP_1} \subset M^\star$.*

(iv) *$M^\star$ is concave: Every point in $M(r_1)$ that is not in $M^\star$ is connected to P_1: $P \in \{M(r_1) - M^\star\} \Rightarrow \{\overline{PP_1} - P_1\} \subset \{M(r_1) - M^\star\}$. Each of the (apart from $\mathbf{a}_1, l_1$) disjoint two parts of $M(r_1) - M^\star$ that result from the bisection according to (i) is convex. This holds also if $(\mathbf{a}_1, l_1)$ is on the boundary of $M(r_1)$.*

(v) *$M^\star$ is symmetric in the sense that any point in the reswitching body, connected to the star-point, is on a straight line, which is wholly in the body, to the extent that the points on the line are semi-positive: $P_0 = (\mathbf{a}_0, l_0) \in M^\star, P(\lambda) = \{\lambda(\mathbf{a}_0, l_0) + (1-\lambda)(\mathbf{a}_1, l_1) \geq 0\} \Rightarrow P(\lambda) \in M^\star$.*

(vi) *The same holds for the complement of $M^\star$ in $M(r_1)$: $P_0 \in M(r_1), P_0 \notin M^\star, P(\lambda) = \{\lambda(\mathbf{a}_0, l_0) + (1-\lambda)(\mathbf{a}_1, l_1) \geq 0\} \Rightarrow \{P(\lambda) - P_1\} \notin M^\star$.*

(vii) *$P_1 = (\mathbf{a}_1, l_1)$ is the only star-point of $M^\star$.*

(viii) *Generalising (ii): if $z_i(r)$ is strictly monotonous in any interval $0 \leq r_a \leq r \leq r_b < R$, there is no reswitching on coordinate axis i in $[z_i(r_a), z_i(r_b)]$ and these points cannot belong to $M^\star$.*

(ix) *The movements of the $z_i(r)$ are restricted by the following relationship, which we call the star-equation:*

$$\frac{a_{11}}{z_1(r)} + \cdots + \frac{a_{1n}}{z_n(r)} + \frac{l_1}{z_{n+1}(r)} = 1; 0 \leq r < R.$$

(x) *The direction of the movement of the $z_i(r)$ is constrained by the derivative of the star-equation (ix):*

$$\frac{a_{11}}{(z_1(r))^2} z_1'(r) + \cdots + \frac{a_{1n}}{(z_n(r))^2} z_n'(r) + \frac{l_1}{(z_{n+1}(r))^2} z_{n+1}'(r) = 0.$$

Proofs and comments: (i) The bisection is not on the bounding of $M(0)$, since it goes through $(\mathbf{a}_1, l_1)$, with $0 < l_1 < z_{n+1}(r_1)$. The intersection of convex sets is convex. (ii) follows from the strict monotonicity of $z_{n+1}(r) = \hat{p}_1(r)$. (iii) $M^\star$ is star-shaped, because $(\mathbf{a}_1, l_1)$ is contained in $M(r)$ for all r. (iv) Points on the line segment contained in $M^\star$ cannot exist because of (iii). If $P_1 \in M(r_1) - M^\star$ and $P_2 \in M(r_1) - M^\star$ are on the same side of the bisection, but there is $P_3 \in M^\star$ on the connecting line segment between them, there must be r_3 such that $P_3 \in M(r_3)$. But, then, $M(r_3)$ separates P_1 and P_2, contrary to the assumption. (v) and (vi) follow from the fact that $M(r_1) \cap M(r_2)$ is a full $(n-1)$-dimensional subspace, only restricted to the non-negative orthant. That $(\mathbf{a}_1, l_1)$ is the only star-point (vii) follows again from the fundamental neo-Ricardian theorem. To see it, suppose there was a second star-point. It could be connected to the first within $M^\star$ because of (v). A line going through both points would have to be in all $M(r)$ in

$\mathbb{R}_+^{n+1}$, hence $\tilde{\mathbf{p}}(r)$ would have to move in the plane determined by the straight line and the origin, which is impossible, if the system is regular. (viii) is proved in the same way as (ii). It may be noted that no reswitching will take place also in the neighbourhood of the coordinate axis, on which reswitching cannot take place, except for flukes. (ix) follows from

$$\sum_{i=1}^{n} \frac{a_{1i}}{z_i} + \frac{l_1}{z_{n+1}} = \sum_{i=1}^{n} \frac{a_{1i}(1+r)\hat{p}_i(r)}{\hat{p}_1(r)} + \frac{l_1}{\hat{p}_1(r)} = \frac{\hat{p}_1(r)}{\hat{p}_1(r)} = 1.$$

This means that, if $z_i(r)$ grows strongly, the other $z_j(r)$ collectively have to contract. (x) is obtained by differentiating (ix). If $n = 2$, because $z'_{n+1}(r) > 0$ and $z'_1(r) < 0$, $z_2(r)$ may have to increase. But the movement according to the star-equation consists broadly in a contraction of $z_1(r), \ldots, z_n(r)$, given the rise of z_{n+1}.

We have seen that the simplex $M(r)$ is spanned by $z_i(r)\mathbf{e}_i$, that is, by points on the coordinate axes that move with r according to functions which depend on the prices $\hat{p}_i(r)$, determined by the system $(\mathbf{A}, \mathbf{l})$. If $n = 2$ and z_{n+1} is on the vertical ordinate, $M(-1)$ is horizontal, $z_{n+1} = l_1$ and $z_1(-1)$ and $z_2(-1)$ are infinite. As r rises from -1 to R, $M(r)$ is a triangle with its tip going up and $z_1(r)$ and $z_2(r)$ getting smaller, until $M(R)$ is vertical. Reswitching will be associated with a non-monotonicity of the intersection of the edges of $M(r_1)$ and $M(r_2)$.

We now want to show that the reswitching body is also spanned by points, but they are not on the coordinate axes in general, but on the semi-positive quadrants of the two-dimensional coordinate hyperplanes H_{ij}; $1 \leq i < j \leq n+1$; of which there are $n(n+1)/2$. Since $\tilde{\mathbf{p}}(r) > 0$, the n-dimensional hyperplane, in which $M(r)$ is contained, will cut every H_{ij} in a line $h_{ij}(r)$ which, restricted to the semi-positive orphant, describes the edge of $M(r)$ between the vertices $z_i(r)$ and $z_j(r)$, on the coordinates i and j, which span H_{ij}. These edges $h_{ij}(r)$ of $M(r)$ are given by the set of points $(\mathbf{a}_0, l_0) = x_{ij}\mathbf{e}_i + y_{ij}\mathbf{e}_j$ fulfilling (7) for given r, and this yields the equation:

$$(1+r)(x_{ij}\hat{p}_i(r) + y_{ij}\hat{p}_j(r)) = \hat{p}_1(r) \tag{9}$$

We thus get a straight line, given r, with y_{ij} as a function of x_{ij}. These lines are well-defined and connect the respective vertices, because all prices are positive. This holds for $1 \leq i \leq j \leq n+1$, since we defined $\tilde{p}_{n+1} = 1$.

We now turn to $M(r_1) \cap M(r_2)$. This set has only the intersections of $h_{ij}(r_1)$ and $h_{ij}(r_2)$ in common in H_{ij}. These intersections follow from equation (7), inserting r_1 and r_2 for r to obtain two equations. The intersections will be denoted as $\mathbf{q}^{ij}(r_1, r_2)$, considered as vectors. The vectors can be written as $\mathbf{q}^{ij} = x_{ij}\mathbf{e}_i + y_{ij}\mathbf{e}_j$. The $h_{ij}(r_1)$ and $h_{ij}(r_2)$ do not necessarily intersect in $\mathbb{R}_+^{n+1}$, but may have negative components or their intersection may diverge to infinity in certain interesting special cases. We write $\mathbf{q}_+^{ij}$, if the components are semi-positive, and we get the interesting proposition:

Proposition. *$M(r_1) \cap M(r_2)$ is spanned by the $\mathbf{q}_+^{ij}$.*

Proof. If P is in the convex hull of the $\mathbf{q}_+^{ij}$, P is in $M(r_1)$ and $M(r_2)$. Conversely, any P in $M(r_1) \cap M(r_2)$ is represented by a vector that fulfils equation (7) for $r = r_1, r_2$. The set of solutions is an $(n-1)$-dimensional manifold, the intersections of which with $\mathbb{R}_+^{n+1}$ is spanned by the extreme points $\mathbf{q}_+^{ij}$.

Hence $M^\star$ is generated by the movement of the $\mathbf{q}_+^{ij}$ with r_2, given r_1, so that the $\mathbf{q}_+^{ij}$ leave a trace on h_{ij}, which is denoted byf_{ij}^+. The $\mathbf{q}^{ij}$ leave a trace on h_{ij} that may extend beyond $\mathbb{R}_+^{n+1}$, which is denoted by f_{ij}. The f_{ij} are connected line segments, except where a $\mathbf{q}^{ij}$ diverges to infinity. $M^\star$ is contained in the convex hull of the f_{ij}^+, where $f_{ij}^+ = f_{ij} \cap \mathbb{R}_+^{n+1}$.

We now express the $\mathbf{q}^{ij}$ in terms of the prices of $(\mathbf{A}, \mathbf{l})$, using (7). We limit the calculations to the case $r_1 = 0$ and $r_2 = \bar{r}$, $0 < \bar{r} \leq R$ for several reasons. There is no loss of generality involved, insofar as choosing the possibilities of starting from a high r_1 and looking for a low r_2 as cases for reswitching and the opposite possibility, choosing a low r_1 and looking for a high r_2, are symmetric; it suffices to consider only one. To choose $r_1 = 0$ has a deeper reason. If one formally considers also negative rates of profit, we saw that $M(-1)$ is a horizontal triangle, if $n = 2$, while $M(R)$ is vertical. An economically relevant transition takes place in the middle at $r = 0$. As we noted above, paradoxes similar to reswitching, but not identical with it, are found, if one looks for intersections between wage curves in the interval $-1 \leq r \leq 0$. Han and Schefold (2006) show that switches in that interval indicate that techniques differ in their sectoral capital-intensity, while the effects become macroeconomic in that they affect the aggregate capital-intensity, if $r > 0$. Studying this transition helps to extend the analysis, if one is interested in the sectoral paradoxes and their connection with reswitching. However, we here focus on the interval $[0, R]$, since the debate has focused on this case. It must be kept in mind in what follows, that reswitching in the exact sense of two switch-points, with the second indicating an increase of the capital-intensity, as the rate of profit is raised beyond the switch-point, requires $r_1 > 0$, whereas $r_1 = 0$ means that the two capital-intensities are equal, precisely because we are dealing with the transition.

Our calculations begin with the most important special case, $i = 1$ and $j = n + 1$, where z_1 is strictly monotonically falling and z_{n+1} strictly monotonically rising ($z_1 = 1/(1 + r), z_{n+1} = \hat{p}_1(r)$). Inserting $\mathbf{q}^{1,n+1} = x_{1,n+1}\mathbf{e}_1 + y_{1,n+1}\mathbf{e}_{n+1}$ in (7) for $r = 0$ and $r = \bar{r}$ results in the equations (omitting subscripts, where not necessary, and writing r for $\bar{r}$):

$$x\hat{p}_1(0) + y = \hat{p}_1(0)$$
$$x(1 + r)\hat{p}_1(r) + y = \hat{p}_1(r).$$

Hence

$$x = \frac{\hat{p}_1(r) - \hat{p}_1(0)}{(1 + r)\hat{p}_1(r) - \hat{p}_1(0)}, y = \frac{r\hat{p}_1(0)\hat{p}_1(r)}{(1 + r)\hat{p}_1(r) - \hat{p}_1(0)}. \tag{10}$$

Clearly, x and y remain positive for $0 < r < R$. The limits are:

$$x(R) = \lim_{r \to R} \frac{1 - \hat{p}_1(0)/\hat{p}_1(r)}{1 + r - \hat{p}_1(0)/\hat{p}_1(r)} = \frac{1}{1 + R}.$$

$$y(R) = \lim_{r \to R} \frac{r\hat{p}_1(0)}{1 + r - \hat{p}_1(0)/\hat{p}_1(r)} = \frac{R}{1 + R}\hat{p}_1(0).$$

Further we get, using the rule of de L'Hospital,

$$x(0) = \lim_{r \to 0} \frac{\hat{p}_1'(r)}{(1 + r)\hat{p}_1(r) + \hat{p}_1'(r)} = \frac{1}{1 + \hat{p}_1(0)/\hat{p}_1'(0)} < 1.$$

$$y(0) = \lim_{r\to 0} \frac{\hat{p}_1(0)[\hat{p}_1(r) + r\hat{p}_1'(r)]}{(1+r)\hat{p}_1'(r) + \hat{p}_1(r)} = \frac{\hat{p}_1(0)}{1 + \hat{p}_1'(0)/\hat{p}_1(0)}.$$

In the case of the labour theory of value, we can use Sraffa's wage curve $w = 1 - r/R$. Standard prices p_i are equal to prices $\hat{p}_i(0)$ at all rates of profit, hence $\hat{p}_i(r) = \frac{R}{R-r}p_i(0)$. We get

$$x(r) = \frac{\hat{p}_1(0)\left(\frac{R}{R-r} - 1\right)}{\hat{p}_1(0)\left(\frac{1+r}{R-r}R - 1\right)} = \frac{1}{1+R}.$$

$$y(r) = \frac{(\hat{p}_1(0))^2 \frac{rR}{R-r}}{\hat{p}_1(0)\left(\frac{1+r}{1+R}R - 1\right)} = \frac{R}{1+R}\hat{p}_1(0).$$

The limits obtained for $x(0)$ and $y(0)$ can in the case of the labour theory of value be shown to be equal to $x(R)$ and $y(R)$, respectively.

For $i = 2, \ldots, n$ and $j = n+1$, one finds that $x(r) > 0$, $0 \le r \le R$, as above, but $y(r)$ may become negative in some range: a fact which we shall have to interpret below. In the case of the labour theory of value, one gets:

$$x = \frac{\hat{p}_1(0)}{\hat{p}_i(0)}\frac{1}{1+R}, y = \frac{R}{1+R}\hat{p}_1(0).$$

No general results seem to exist for the remaining cases $1 \le i < j \le n$, except if the labour theory of value holds: then, x and y will diverge.

4 Illustration by Means of Diagrams

We illustrate the preparatory results geometrically and empirically. If $n = 2$, and if we stick to the assumption $r_1 = 0$ and look for switch-points in $0 < r < R$, we get a three-dimensional diagram. If the labour theory of value holds, $\mathbf{q}^{13} = [1/(1+R), R\hat{p}_1(0)/(1+R)]$ and $\mathbf{q}^{23} = [\hat{p}_1(0)/(1+R)\hat{p}_2(0), R\hat{p}_1(0)/(1+R)]$ and $\mathbf{q}^{12}$ diverges. The simplex $M(r)$ stands up, starting from $M(0)$, turning around an axle fixed by $\mathbf{q}^{13}$ and $\mathbf{q}^{23}$, representing $M^\star$ and containing $P_1 = (\mathbf{a}_1, l_1)$. The lines h_{12} remain parallel, for $\mathbf{q}^{12}$ diverges, as we saw above (Figure 1):

If $n = 2$, the variation of relative prices of a regular system with r will cause the simplex to turn in some way such that $M(0) \cap M(r)$ always contains $(\mathbf{a}_1, l_1)$ – $M^\star$ is starshaped. Note that $\mathbf{l} > 0$ so that $(\mathbf{a}_1, l_1)$ is always above $h_{12}(r)$. $M(r)$ turns vertical at $r = R$. As we saw, $z_1(r) = 1/(1+r)$ will always fall and $z_2(r) = \hat{p}_1(r)/(1+r)\hat{p}_2(r)$ will also tend to do that; the factor $(1+r)$ helps to get this effect (see Section 2). The likeliest case seems to be (it is at any rate the one most frequently observed in our empirical investigation, see below) that x_{13} falls, while x_{23} rises; the latter is possible with $z_2(r)$ falling, if $\hat{p}_1(r)$ goes up fast enough. $M^\star$ will then cover a larger area of $M(0)$; see Figure 2, where z_2 is not monotonous. As Ricardo realised in 1815 (Ricardo, 1951), the price of

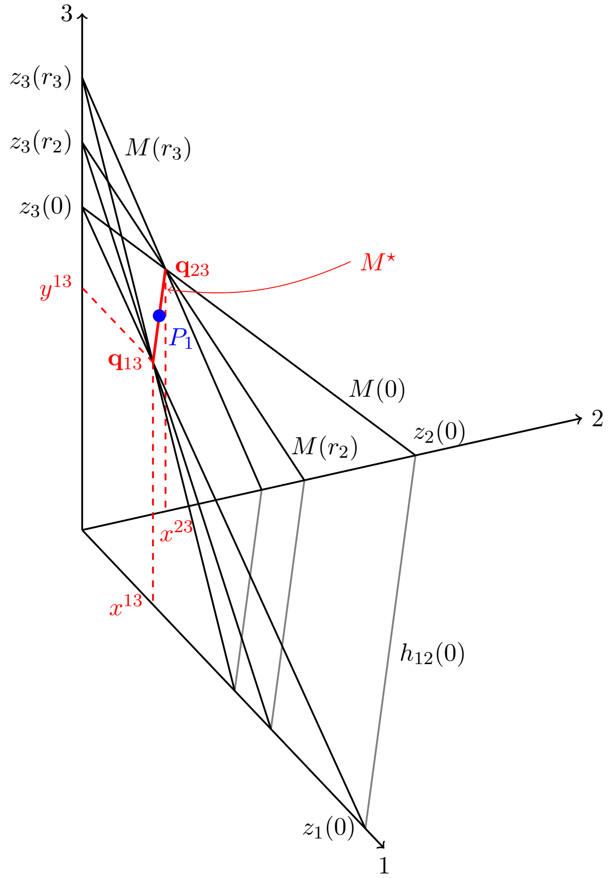


Figure 1: Labour theory of value case

the product of a capital-intensive industry rises relative to other prices with a rise of the rate of profit. Hence it seems that a high capital-intensity in sector 1, the one examined for reswitching, is favourable for it, and this will be confirmed in the next example.

Prices in terms of wage rate and Sraffa's standard prices do not deviate from linearity as much and as often as once had been thought both for theoretical reasons and according to empirical findings summarised in Section 2. However, $z_2(r)$ need not always be falling, as in Figure 2, and it need not be monotonous. Figure 3 illustrates the possibility that $z_2(r)$ increases in some interval, so that x_{23} increases as well, with $y_{23} < 0$ but $y_{13} > 0$, and $y_{12} > 0$.

We speak of a transgression on coordinate i, if $z_i(r)$ rises beyond $z_i(0)$ in such a way that $\mathbf{q}^{12} > 0$. A transgression on one coordinate means that there are regressions on others according to star-equation (ix), *Lemma, Properties of the Reswitching Body*, above.

Even without transgressions, our geometrical analysis has shown:

Proposition. *If $n = 2$, if the system is regular and if $(\mathbf{a}_1, l_1) > 0$, the reswitching body consists of two triangles with vertices meeting at $(\mathbf{a}_1, l_1) > 0$ with equal angles, and*

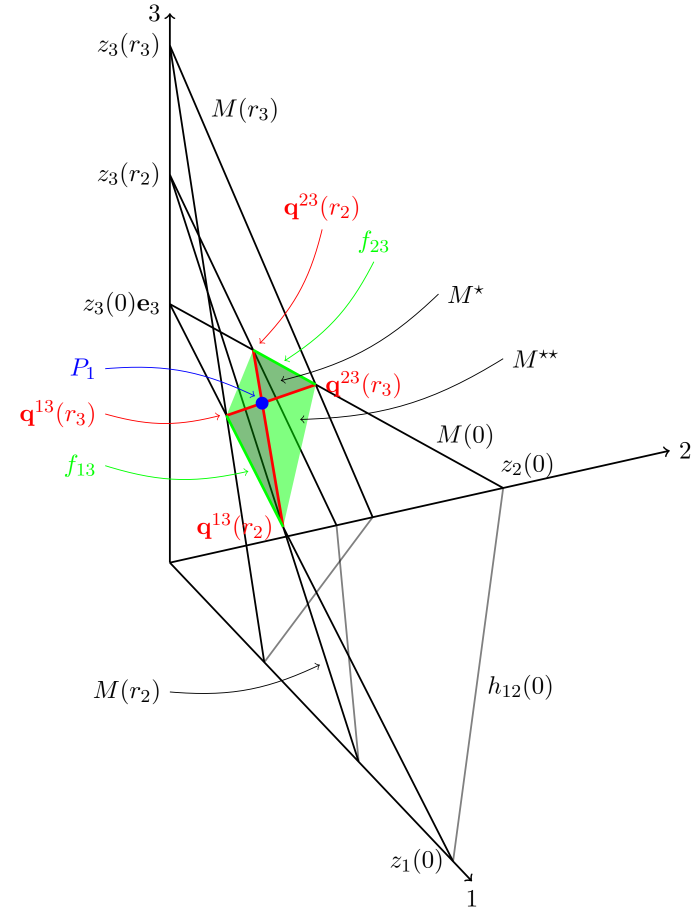


Figure 2: $M^\star$ (dark green area) is contained in the convex hull $M^{\star\star}$ (light green area) of f_{13} and f_{23} and this is contained in a simplex $M^{\star\star\star}$, spanned by $\bar{\mathbf{f}}_{13} = \mathbf{q}^{13}(r_2), \bar{\mathbf{f}}_{23} = \mathbf{q}^{23}(r_3)$ and $z_3(0)\mathbf{e}_3$, to be considered in *Theorem 1* below. The vertices 1 and 2 move toward the origin, 3 away from it (Lemma ix). Accordingly, $z_1(r)$ and $z_2(r)$ are falling, so is x_{13}, but x_{23} increases, as the line segment $M(0) \cap M(r)$ turns around $P_1 = (\mathbf{a}_1, l_1)$ and z_2 is not monotonous. Note that $\mathbf{q}^{12}$ is obtained as the point where a line through $\mathbf{q}^{23}$ and $\mathbf{q}^{13}$ hits the plane H_{12} and intersects with $h_{12}(0)$ outside the non-negative orthant (not drawn).

$\mathbf{q}_{13}(r_1, r_2)$ *and* $\mathbf{q}_{23}(r_1, r_2)$ *move with* r_2 *in opposite directions, given* r_1.

We change over to higher dimensions. As a preliminary, it is possible to give a visual representation of the reswitching body $M^\star$ in the case $n = 3$, for, although the 4 vertices of $M(r)$ are then on 4 coordinate axes in 4-dimensional space, $M(r)$ is a three-dimensional body and $M(r_1) \cap M(r_2), r_1 \neq r_2$, is a two-dimensional intersection, because the movement of $\tilde{\mathbf{p}}(r)$ is never enclosed in any subspace. Figure 4 shows $M(0)$ as a tetrahedron, cut by a plane, the cut representing $M(0) \cap M(r)$. The movement of $M(r)$ with r results in a reswitching body indicated by the green area. It is assumed that prices deviate only

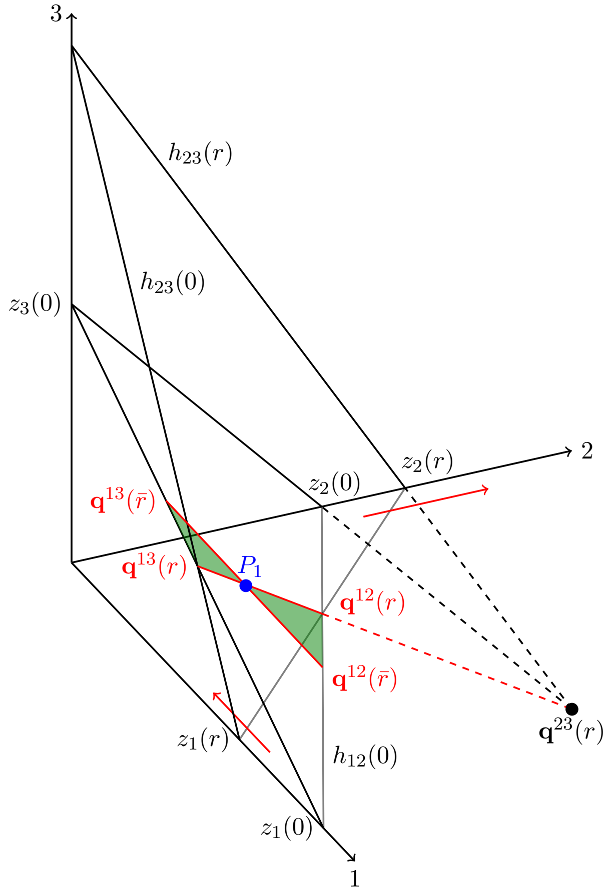


Figure 3: The case of a transgression. Note that $\mathbf{q}^{23}$ is in H_{23}, with $y_{23} < 0$, $x_{23} > 0$, in the point of H_{23} where the three lines $h_{23}(0), h_{23}(r)$ and the straight line through $\mathbf{q}^{13}$ and $\mathbf{q}^{12}$ meet. The green area illustrates how $M^{\star}$ might grow with a further increase of r to $\bar{r}$.

moderately from the labour values. This implies that the cut is such that the semi-positive $\mathbf{q}_{+}^{ij}$ are on the three edges h_{14}, h_{24}, h_{34}, spanning $M(0) \cap M(r)$, the other $\mathbf{q}^{ij}$ being outside $\mathbb{R}_{+}^{4}$ (Figure 4).

If prices are close to labour values, the reswitching body is near the tip, representing labour, of the tetrahedron, while a transgression would take place at one of the vertices of the basis.

We have two possibilities, each limited, to represent systems with large n in such a way that our insights from the three-dimensional example remain useful. If $(\mathbf{a}_0, l_0) \in M^{\star}$, $(\mathbf{a}_0, l_0)$ will fulfil (2) for some $r = r_1$ and $r = r_2 \neq r_1$, and this may be written as, with

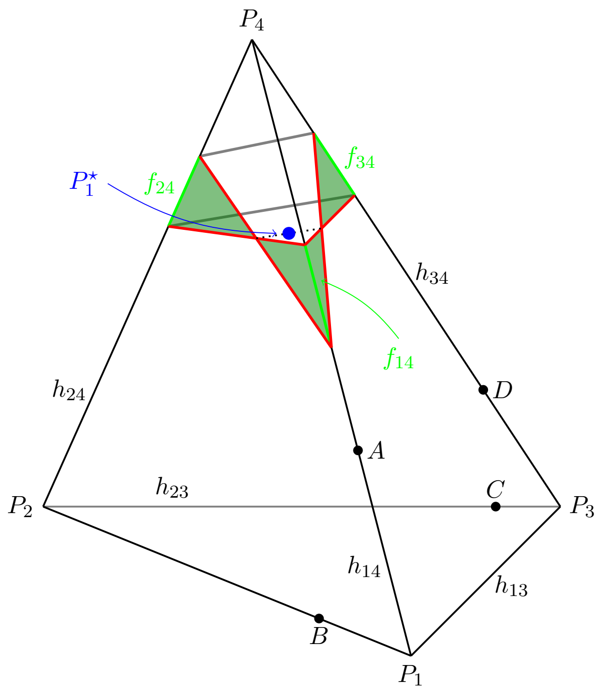


Figure 4: If $n = 3$, $M(0)$ becomes a tetrahedron, spanned by the vertices P_1, P_2, P_3, P_4 on the four coordinate axes. $M(0) \cap M(r)$ results from a cut as a triangle, if one vertex (P_4 for labour) is above the plane, three others (for the commodities) below. A cut with four corners A, B, C, D is also possible, with two vertices on each side of the separating hyperplane. The reader may here verify (i) – (x) of the Lemma, in particular the two-sided concavity of $M^\star$ and the convexity of the two parts of the bisected tetrahedron. The star-point is here denoted $P_1^\star$. $M^\star$ appears as bordered by straight lines, because we have drawn only two intersecting two-dimensional hyperplanes. The geometry is somewhat more complicated and the concave surfaces get smooth, if more hyperplanes are drawn, but we do not go into the details for reasons of space.

$r = r_1, r_2$:

$$a_{01}(1+r)\hat{p}_1(r) + a_{0i}(1+r)\hat{p}_i(r) + l_0 = \hat{p}_1(r) - \sum_{j \neq i} a_{0j}(1+r)\hat{p}_j(r) = \phi_i(r). \qquad (11)$$

The formula shows that the insights gained from examples with $n = 2$ can be extended to $n > 2$ under certain conditions. We have visualised how $M^\star$ is situated in $M(r_1)$, that is, how likely it is that a method with a switch at $r = r_1$ generates a second switch at some r_2. The reswitching possibilities were seen to depend on the rates of change of prices and of the $z_i(r)$, for instance. The constellations will remain essentially the same, if $\phi_i(r)$ is monotonically rising, and this will be the case, if we replace $(\mathbf{a}_0, l_0)$ in the formula above by $(\mathbf{a}_1, l_1)$, for then the left-hand side will rise monotonically, hence also $\phi_i(r)$. So we get an approximation for all $(\mathbf{a}_0, l_0)$ in the vicinity of $(\mathbf{a}_1, l_1)$.

The other possibility is to restrict the analysis to vectors with $a_{0j} = 0, j \neq i$, using the prices of the n-dimensional system and thus transcending a three-dimensional model. For this, an empirical investigation has been undertaken.

5 An Empirical Investigation

The prices of a system $(\mathbf{A}, \mathbf{l})$ have been calculated, taking as the system an input-output matrix for Germany of the year 2011, using a data set published by Zambelli as supplementary data for Zambelli (2018). The calculations were made by Jakob Kalb (Kalb, 2021), to whom I owe thanks. Prices with a uniform rate of profit can be calculated, as has been done in the last four decades. Section 2 refers to that literature and discusses its main result: prices as a function of the rate of profit do vary with distribution, but, if expressed in terms of labour commanded or as Sraffa's standard prices, they turn out to be often quasi-linear, mostly monotonous and inflection points are not frequent. Section 2 mentions proposals to give more precision to such characterisations and to explain them. Here, we are not concerned with the prices directly, but with the reswitching body, which ultimately depends on the movement of the vertices of $M(r)$, hence on the $z_i(r); i = 1, \ldots, n+1$; where, as above,

$$z_i(r) = \frac{\hat{p}_1(r)}{(1+r)\hat{p}_i(r)}; i = 1, \ldots, n; z_{n+1}(r) = \hat{p}_1(r).$$

All 32 curves $z_i(r)$ were calculated and represented in diagrams. In accordance with our results of Section 2 and with the geometrical properties of the reswitching body, a clear majority of curves fall monotonically and are convex. Only one curve is not convex, and five curves rise after an initial fall.

Given the $z_i(r)$, it is elementary to deduce the edges $h_{ij}(r)$ of $M(r)$ and the intersections $\mathbf{q}^{ij}$ in each coordinate hyperplane H_{ij}. The H_{ij} are numerous. Relevant are the $H_{1,n+1}, \ldots, H_{n,n+1}$, for, as above, the x-component of $\mathbf{q}^{ij} = x_{ij}\mathbf{e}_i + y_{ij}\mathbf{e}_j$ is positive for all i, if $j = n+1$. Hence $x_{i,n+1}(r)$ was calculated from (10) for $i = 1, \ldots, n$. In addition, $t_i = x_{i,n+1}(r)/z_i(0)$ was calculated for $i = 1, \ldots, n$ in order to analyse potential cases of transgression, and only in cases of transgression, the $H_{ij}, j \neq n+1$, play a role, as we shall see (R3). Results:

R1. Of 31 curves for t_i, only 4 are not monotonous, 19 are monotonically falling and 8 rising. Three cases of transgression were found in that t_i rises above one, as in Figure 3. The variation of t_i was less than 10% in 19 of 31 cases. These results were obtained when treating the sector *Agriculture, hunting, forestry and fishing* – which had been listed first in Zambelli's data set – as the first sector, i.e. as the sector in which reswitching is examined.

R2. The calculations were repeated and extended for a German input-output matrix of 2014, which had previously been used for the *two-country* and *five-country cases* in Kersting and Schefold (2021), based on the 2016 Release of the *World Input Output Database*. After adjustment, t_i was calculated for 54 sectors. 31 curves were falling, 18 rising and only 5 not monotonous. Transgression occurs in 10 of 54 cases, in that

t_i rises above one. In a second round of calculations, the first sector (which had happened to be *Crop and animal production, hunting and related service activities*) was exchanged with a capital-intensive sector (capital-intensity measured in labour values) *Manufacture of basic metals*, which therefore now produced the first good and became the one for which reasons for a low or high likelihood of reswitching were being sought. Here, t_i rose above one in 39 out of 54 cases. In 15 of the cases, $t_i > 1$ occurred even for $r = 0$, and this happened, if $z_i(r)$ was monotonically rising, implying that $\hat{p}_1(r)$ rose relatively to $(1+r)\hat{p}_i(r)$ – a phenomenon characteristic for capital-intensive production according to Ricardo's observation referred to above. If, however, the sector *Financial service activities, except insurance and pension funding* was promoted to the role of sector one, only 2 in 54 cases resulted in a rise of t_i above one. Also, the majority of curves now were falling (47 out of 54).

The results reported so far are general and are not confined to the use for a visualisation in three dimensions; however, they will now be used for this purpose by calculating $f_{1,n+1}, f_{i,n+1}$ and $f_{1,i}$ in three dimensions. These line segments are projected into the plane $H_{1,i}$. The convex hull of the projected line segments is called $C_i; i = 2, \ldots, n$. A vector $(\mathbf{a}_0, l_0)$ of the form $(a_{01}, 0, \ldots, 0, a_{0i}, 0, \ldots, 0, a_{0n}, l_0)$ is in $M^\star$ and generates reswitching only, if its projection is in C_i (Figure 5).

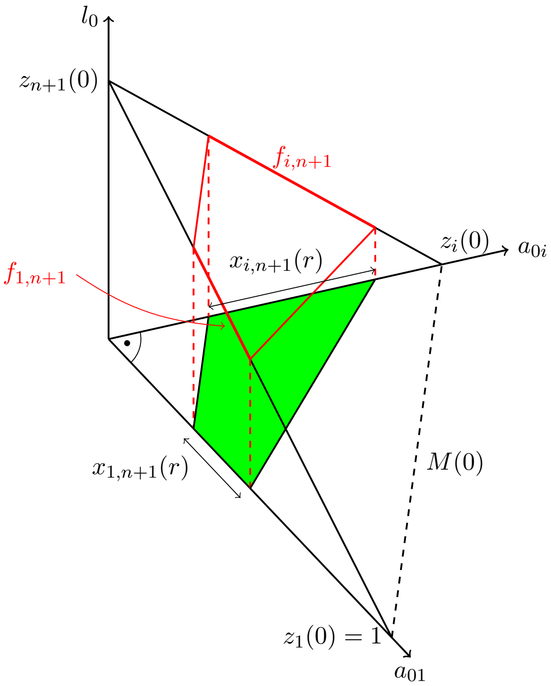


Figure 5: Projection of the reswitching body with its convex hull according to R2 above.

This is a necessary condition, but not sufficient because, by taking the convex hull, we have covered the concavities of $M^\star$. Going from 1 to n, we get by projecting in $H_{1,2}, H_{1,3}, \ldots, H_{1,n}$ $(n-1)$ pictures of the three-dimensional surface of the convex hull of $M^\star$. The analysis could be extended to the interior of $M^\star$ by calculating $\phi_i(r)$ and using formula (11), but this has so far only been considered in theory. By contrast, the C_i have been calculated and drawn by Jakob Kalb from the derived data on prices, using the input-output table with 31 sectors of 2011 mentioned above and used for R1. Now we get R3:

R3. We now return to the calculations for the case of 31 sectors, based on Zambelli's data of the year 2011. If *Agriculture, hunting, forestry and fishing* is used as sector one, the convex hulls are visibly very narrow and reflect the fact that the projected variations of $\mathbf{q}^{1,n+1}$ and $\mathbf{q}^{i,n+1}; i = 2, \ldots, n$; are very narrow, and this means that surfaces, of which the hulls $C_2, \ldots, C_n$ are projections, must be very narrow. The conclusion is valid, since $M_i(0)$ is very close to the plane, because $\hat{p}_1(r)$ rises very slowly, except near $r = R$, as can be seen from the corresponding price diagram. In all but two cases, the projected line segments of $f_{1,n+1}$ and $f_{i,n+1}$ are quite short, and $f_{1,i}$ is empty, because $\mathbf{q}^{1,i}(r)$ is not semi-positive. Only two cases are different and show a transgression, implying that a short stretch of $f_{1,i}$ appears (Figures 6 and 7).

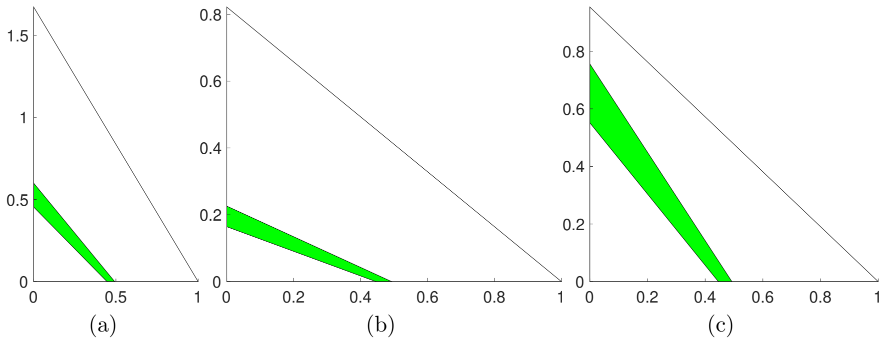
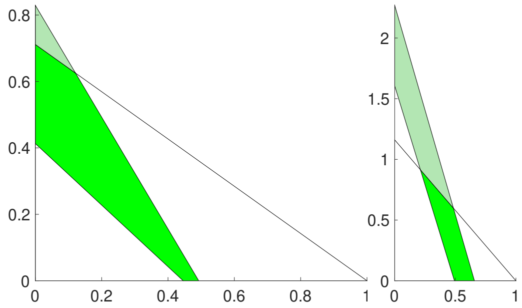


Figure 6: Three representative cases without transgression, designated as C_2 (a), C_8 (b) and C_{19} (c) in Kalb (2021). In each case, the abscissa shows the interval between 0 and $z_1(0) = 1$, while the ordinate stretches from 0 to $z_i(0)$ for $i = 2, 8, 19$.

It turns out that *Coke, Refined petroleum and Nuclear fuel* is the most capital-intensive sector. If it is chosen as sector one, ten cases of transgression are found and the projected convex hulls are a little less narrow in the 21 other cases. Qualitatively, the pictures are quite similar. Finally, if the least capital-intensive sector, *Sale, Maintenance and Repair of Motor Vehicles and Motorcycles*, is taken as sector one, the projected convex hulls become very narrow in most cases and no transgression occurs.



(a) (b)

Figure 7: Left diagram: One of only two cases of transgression, which appear, if *Agriculture, hunting, forestry and fishing* is sector 1: C_{21} (a). On the right, one of ten transgressions is represented, if *Coke, Refined petroleum and Nuclear fuel* is chosen as sector 1: C_{19} (b).

6 Five Theorems on the Probability of Reswitching

Our findings justify the assumptions of the following theorem:

Theorem 1. *Assumption (A): Let a productive indecomposable system* $(\mathbf{A}, \mathbf{l})$ *be given,* $\mathbf{A} \geq 0, \mathbf{l} \geq 0, (\mathbf{a}_1, l_1)$ *being the first process. An alternative process* $(\mathbf{a}_0, l_0)$ *is as profitable as process* $(\mathbf{a}_1, l_1)$ *at* $r = 0$*. The probability as defined above for the existence of a reswitch for* $(\mathbf{a}_0, l_0)$

(1.1) is zero, if the labour theory of value holds,

(1.2) is positive, if the system is regular,

(1.3) tends to zero with $n \to \infty$ *for* r *in* $[\epsilon, \bar{r}], 0 < \epsilon < \bar{r} < R$*, if the system is regular and if the movement of relative prices is bounded by the condition*

$$\frac{p_1(r)}{p_i(r)} < (1+\epsilon)\frac{p_1(0)}{p_i(0)} \qquad (\star)$$

for $i = 2, \ldots, n$*.*

Proof. (1.1) and (1.2) were proved in Schefold (1976); the main arguments have been repeated here. As to (1.3), the condition, rewritten as

$$\frac{p_1(r)}{p_1(0)} < (1+\epsilon)\frac{p_i(r)}{p_i(0)} \leq (1+r)\frac{p_i(r)}{p_i(0)},$$

implies

$$\frac{\hat{p}_1(r)}{\hat{p}_1(0)} - 1 < (1+r)\frac{\hat{p}_i(r)}{\hat{p}_i(0)} - 1,$$

hence

$$\frac{\hat{p}_1(r) - \hat{p}_1(0)}{(1+r)\hat{p}_i(r) - \hat{p}_i(0)} < \frac{\hat{p}_1(0)}{\hat{p}_i(0)}.$$

The left-hand side is equal to $x_i(r)$ according to (10), extended from $i = 1$ to $i = 2, \ldots, n$, the right-hand side to $z_i(0)$. The $x_i(r)$ are continuous in $[\epsilon, \bar{r}]$ and have a maximum $\bar{x}_i$. This means that the $f_{i,n+1}$ extend on $h_{i,n+1}$ up to maxima $\bar{f}_i$; the corresponding vector is $\bar{\mathbf{f}}_i$. The $\bar{\mathbf{f}}_i$, together with $z_{n+1}\mathbf{e}_{n+1}$, span a simplex $M^{\star\star\star}$ that contains the tip of $M(0)$, $M^\star$ and its convex hull $M^{\star\star}$ (compare Figure 2) and is contained in $M(0)$. Define, using the euclidean vector norm $\|\cdot\|$, $\|\bar{\mathbf{f}}_i - z_{n+1}\mathbf{e}_{n+1}\|/\|z_i\mathbf{e}_i - z_{n+1}\mathbf{e}_{n+1}\| = \gamma_i$. Clearly, $\gamma_i < 1$. Introduce a coordinate system and the euclidean metric in the n-dimensional hyperplane H_0 containing $M(0)$, let the points corresponding to $z_i\mathbf{e}_i; i = 1, \ldots, n+1$; be represented by vectors $\mathbf{v}_i$ in H_0. The n-dimensional measure μ of $M(0)$ then is

$$\mu(M(0)) = \frac{1}{n!}\left|\det(\mathbf{v}_1 - \mathbf{v}_{n+1}, \ldots, \mathbf{v}_n - \mathbf{v}_{n+1})\right|.$$

We denote the corresponding vectors in H_0 spanning $M^{\star\star\star}$ by $\bar{\mathbf{v}}_i$. The $\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_{n+1}$ are shorter than the $\mathbf{v}_i - \mathbf{v}_{n+1}$ by the factors $\gamma_i; i = 1, \ldots, n$; so that

$$\mu(M^{\star\star\star}) = \frac{1}{n!}\left|\det(\bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_{n+1}, \ldots, \bar{\mathbf{v}}_n - \bar{\mathbf{v}}_{n+1}\right| = \frac{1}{n!}\left|\det(\gamma_1(\mathbf{v}_1 - \mathbf{v}_{n+1}), \ldots, \gamma_n(\mathbf{v}_n - \mathbf{v}_{n+1}))\right|.$$

Hence $\mu(M^{\star\star\star}) = \gamma_1 \cdot \ldots \cdot \gamma_n \mu(M(0))$, and since $\mu(M^{\star\star\star})$ contains $M^\star$, the probability for reswitching $\dfrac{\mu(M^\star)}{\mu(M(0))}$ is at most $\gamma_1 \cdot \ldots \cdot \gamma_n$ and tends to zero for $n \to \infty$. QED.

On the one hand, the theorem overstates the conditions necessary for the probability of reswitching to tend to zero for $n \to \infty$, because the estimate is based on the convex hull of $M^\star$ and does not take the concavity of the reswitching body into account, which is like a concave lens, very thin (ultimately one point) in the middle, but potentially thick on the margin around it. On the other hand, the assumptions of the theorem exclude strong variations of prices, which lead to transgressions. If $x_i(r) > z_i(0)$, $M^\star$ is still contained in $M(0)$, because the analysis is confined to semi-positive $(\mathbf{a}_0, l_0)$. But points exist that are like reswitch-points in that they belong to the hyperplanes containing $M(0)$ and $M(r)$, for some r, $0 \leq r \leq R$. They could be called pseudo reswitch-points; they are not contained in simplex $M^{\star\star\star}$ that contains the tip of $M(0)$, $M^\star$ and the convex hull $M^{\star\star}$ of $M^\star$.

In order to include transgressions in our analysis, we begin with a heuristic description of the possible constellations. $M(0)$ is bisected by $M^{\star\star}$ into the top part M_t (defined as the top because it includes the tip, where there is no reswitching), and the bottom part M_b between $M^{\star\star}$ and the basis B (the top may also reach down to the basis, as we shall see). One shows as in the *Lemma* that M_t and M_b are disjoint and convex and not empty. It is also clear that they are not of measure zero for any given finite n. The basis B is an $(n-1)$-simplex spanned by the endpoints of $z_i(0)\mathbf{e}_i$; $i = 1, \ldots, n$; which coincide with the endpoints of $\mathbf{f}_i$, the vectors that go from the tip of $M(0)$, $z_{n+1}\mathbf{e}_{n+1}$, to $z_i(0)\mathbf{e}_i$. Under the assumptions of *Theorem 1*, there is no reswitching at basis B. Transgressions imply that neighbourhoods, possibly edges and entire simplices of lower order contained in B become part of the reswitching body.

We distinguish two kinds of transgression at vertex i; $i = 2, \ldots, n$; a weak transgression, if $f_{i,n+1}$ contains the vertex $z_i(0)\mathbf{e}_i$, and a strong transgression, if $f_{i,n+1}$ lies entirely beyond the vertex on $h_{i,n+1}$ (Figure 7 left and Figure 7 right). We call a vertex empty, if there is no transgression (Figure 6). Note again that vertices 1 and $n+1$ are always empty. Figure 8 illustrates the three possibilities for the vertices in the case $n = 3$, with the convex hull $M^{\star\star}$ of $M^{\star}$. Compare with Figure 4, where all vertices are empty.

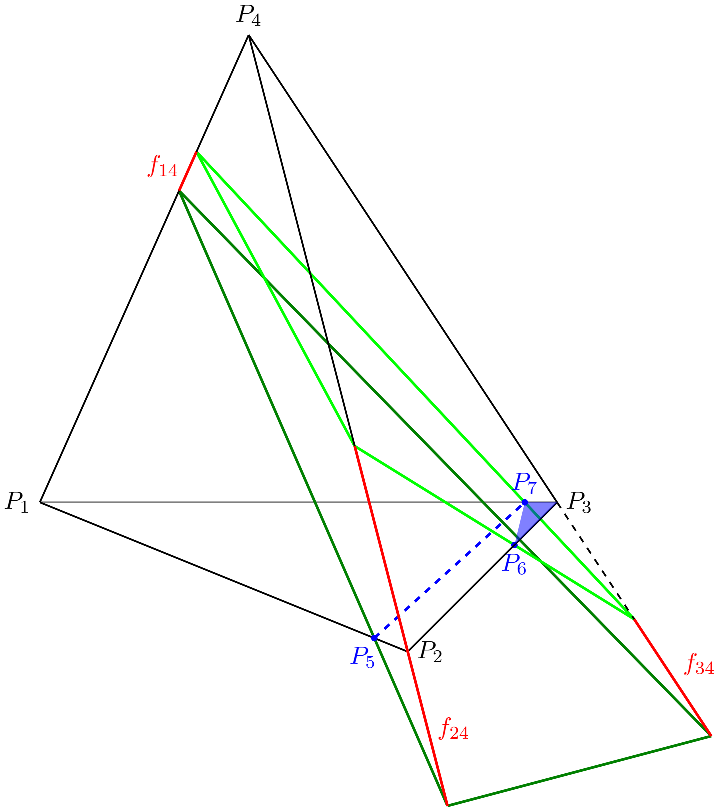


Figure 8: The convex hull $M^{\star\star}$, if $n = 3$ and vertices 1 and 4 are empty, vertex 2 shows a case of a weak and vertex 3 of a strong transgression.

The interested reader may draw such diagrams in the case $n = 3$ for all conceivable constallations, like two empty vertices and one weak transgression, but not all are economically possible. Three transgressions simultaneously are excluded, because vertex one is always empty (we proved in Section 3 that $x_{1,n+1}(r) < z_1(0) = 1$ for $0 \le r \le R$). Note that if we could have strong transgressions at all three vertices of the basis, $M^{\star\star}$ would be empty because the convex combinations of f_{14}, f_{24}, f_{34} would entirely lie outside $\mathbb{R}^4_+$ ("below" the basis $P_1P_2P_3$). By contrast, the probability of reswitching would tend to one, if we could have weak transgressions on *all* vertices P_1, P_2 and P_3 at the basis. In the present case of Figure 8, the transgressions "induce" a boundary of $M^{\star\star}$ on the basis near P_2 and P_3, bordered by $\overline{P_5P_2}, \overline{P_2P_6}, \overline{P_6P_7}$ and $\overline{P_7P_5}$. Again, if the mass of reswitch-points on and above corresponding to this area becomes large enough as n tends to infinity, the probability of reswitching will not go to zero as in *Theorem 1.* At the same time, it is not true anymore that the top is separated from the basis. Area $P_6P_3P_7$ is part of the top M_t in Figure 8.

It is clear from simplex theory that there are hyperplanes of order $n-1$ which bisect $M(0)$ and $M^{\star\star}$ in such a way that all empty vertices are on one side and all vertices with transgressions on the other (compare points A, B, C, D in Figure 4). It is intuitive that, with the method employed in the proof of *Theorem 1*, the measure of that mass of points of $M^{\star\star}$ which is in the upper part will be zero relatively to that of $M(0)$, but that cannot be said for the lower part. The question comes up, addressed in Schefold (2016), whether there is empty room in the upper part of $M(0)$ in this new division to accommodate the points near the basis of $M^{\star\star}$ with the result that $M^{\star\star}$, partly shifted into the empty spaces, will be contained in a simplex like $M(0)$, with only the $\mathbf{f}_i$ shortened.

Whether this will be possible depends highly on the number of empty vertices relative to all vertices. Our empirical investigation has shown (for details see Section 5) that the $z_i(r)$ are predominantly monotonous and transgressions rare on average, in that they are rare if the capital-intensity of sector 1 corresponds to the average or is lower than that. Since we are here concerned with the the probability of reswitching in models with many sectors, the probability has to be calculated for the average case. The number of empty vertices is more than half the number of all vertices, and this is what we should expect also on theoretical grounds: on the one hand because prices tend to be quasi-linear (as we saw in Section 2), on the other, more directly, because Sraffa prices, by virtue of the normalisation, must in part go up, in part down. Since $z_i(r) = p_1(r)/(1+r)p_i(r)$, the factor $1/(1+r)$ implies that the downward pressure prevails. This has been shown in detail in Section 2 for those z_i for which $p_1(r)$ and $p_i(r)$ can be approximated by equation (4) or (5) with $T = 0$ – the majority of price curves according to the empirical literature quoted in Section 2 – and it is in accordance with (ix) and (x) of the *Lemma*. If the majority of the $z_i(r)$ are falling, compensating for the rise of $z_{n+1}(r)$, the majority of the vertices must be empty. The conclusion is essential for what follows, in that it motivates the key formal assumptions of theorems 2-5.

The vertices can be numbered so that vertices $1, \ldots, m$ are empty and $m+1, \ldots, n$ show transgressions. We suppose that $m/n > 1/2$ and that this ratio tends rather to rise than to fall with n, because larger systems will have more random properties. Of course, these are so far only broad tendencies, which can be given a more precise meaning only in specific circumstances – what is needed is a theory of partially random matrices.

We simplify the presentation by using the diagrammatic technique developed for the case $n = 2$, but interpret the edges of $M(0)$ as pertaining to a simplex of higher dimension n. Also, we draw $M^\star$ as if the $(n-1)$-dimensional surfaces bordering it were flat (cf. the remark made in the legend to Figure 4). These surfaces have $(\mathbf{a}_1, l_1)$ in common. We represent them schematically as line segments intersecting at $(\mathbf{a}_1, l_1)$.

We draw the following diagrams assuming that there is an empty vertex on the left and a vertex with transgression on the right; the drawing of the other possible constellations is left to the reader.

We have generally assumed that the system is basic and explained our reasons for doing so, but non-basics have played an important role in an earlier phase of the debate about capital theory, and, anyway, $(\mathbf{a}_1, l_1)$ is not necessarily positive, it may fall on the boundary and even on just one edge of $M(0)$. If f_{14} in Figure 8 shrinks to a point, the first commodity is basic and commodities 2 and 3 non-basic. Non-basics can therefore be considered as limit cases of our analysis, as will be further exemplified in a *Note* at the end of this section. Non-basics do not affect our conclusions substantially, in particular with regard to the following *Theorem 2*.

The simplest estimate of the potential of reswitching is obtained if one includes not only $M^\star$ and $M^{\star\star}$, but also the top of $M(0)$ and the convex hull of all points of reswitching, including those that are not semi-positive ('pseudo switch-points') and that arise because of transgressions as in Figure 7, left side. This set is now called $M^{\star\star\star}$. As in the proof of *Theorem 1*, $\bar{\mathbf{f}}_i$ is the vector going from the tip of $M(0)$ up to the maximum $\bar{f}_i$ of $f_{i,n+1}$, as shown in the schematic Figure 9. Obviously, $\bar{\mathbf{f}}_i = \gamma_i f_i$ with $\gamma_i < 1$, if vertex i is empty and $\gamma_i > 1$, if there is a transgression. The intermediate case $\gamma_i = 1$ is taken up in *Theorem 3*.

Theorem 2. *Suppose a system fulfils assumption (A) of* Theorem 1*, and suppose m is the number of empty vertices and $n - m$ that of transgressions, with $m/n > 1/2$. Suppose the vertices $i = 1, \ldots, n$ can be grouped into $n - m$ not overlapping groups $i_{j1}, \ldots i_{jk}$; $k \geq 1$; $j = m + 1, \ldots, n$; $\sum_{j=m+1}^{n} i_{jk} = m$ with $\gamma_{i_{j1}} \cdot \ldots \cdot \gamma_{i_{jk}} < 1/\gamma_j$. Then the probability that such systems exhibit reswitching will tend to zero as $n \to \infty$.*

The *Proof* is obvious: one gets $\mu(M^{\star\star\star})/\mu(M(0)) \to 0$, and $\mu(M^\star) < \mu(M^{\star\star\star})$. The essential assumption is that the γ_i can be grouped so that for each j with $\gamma_j > 1; j = m + 1, \ldots, n$; there is at least one $\gamma_{i_{j1}}, \ldots, \gamma_{i_{jk}}$ such that $\gamma_{i_{j1}} \cdot \ldots \cdot \gamma_{i_{jk}} \cdot \gamma_j < 1$: the edges with $\bar{f}_i < f_i; i = 1, \ldots, m$; compensate for the edges with $\bar{f}_j > f_j; j = m+1, \ldots, n$. This is plausible if $m >> n - m$. The estimate overstates the probability, because $M^{\star\star\star}$ includes the top and, more importantly, $n - m$ volumina of the 'tails' with not semi-positive points of the transgression, and moreover, points due to the formation of the convex hulls. But the calculation of the volume of $M^{\star\star\star}$ is easy, following the method used in the proof of *Theorem 1*.

The basis B of the n-simplex $M(0)$ consists of an $(n - 1)$-simplex formed from the n tips of the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$. We also can define a $(n - 1)$-simplex S from the tips of the vectors $\bar{\mathbf{f}}_1, \ldots, \bar{\mathbf{f}}_m, \mathbf{f}_{m+1}, \ldots, \mathbf{f}_n$, connecting the maxima of the $f_{i,n+1}$, where the vertices are empty, with the vertices $\mathbf{f}_j$, where there are transgressions, schematically represented in Figure 10. Simplex S divides $M^\star$; a domain D_1 between S, B and the lower boundary of $M^\star$ (represented schematically as flat in the diagram, though it is not) and a domain D_2 between S and the lower boundary of $M^\star$, where $M^\star$ is concave and contains $(\mathbf{a}_1, l_1)$ – now supposed to be positive – as the star-point. Define $M^{\star\star\star}$ this time as the convex hull of the tip of $M(0)$ and S.

Theorem 3. *If $\mu(D_1) \leq \mu(D_2), \mu(M^{\star\star\star})/\mu(M(0)) \xrightarrow[n \to \infty]{} 0$ and the probability of reswitching tends to zero.*

The *Proof* is again obvious; we have

$$\mu(M^{\star\star\star})/\mu(M(0)) = \gamma_1 \cdot \ldots \cdot \gamma_m.$$

Instead of compensating for $\gamma_{m+1} > 1, \ldots, \gamma_n > 1$ as in the previous theorem, we have shifted part of $M^\star$, as Figure 10 illustrates.

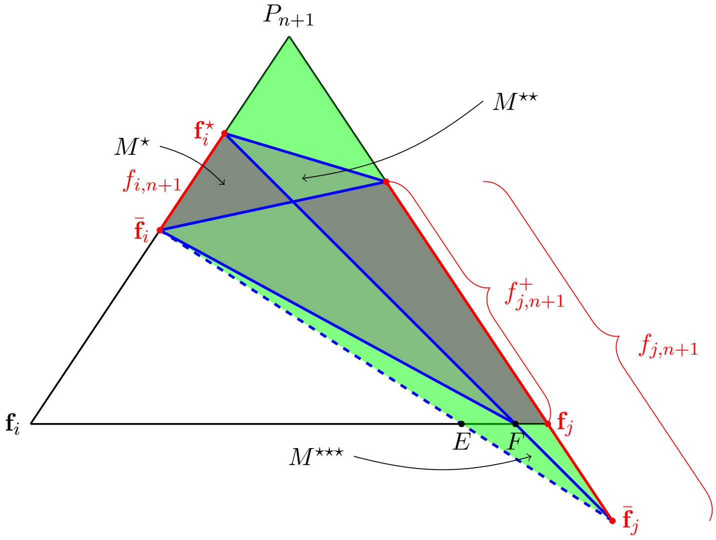


Figure 9: $M^{\star\star\star} \supseteq M^{\star\star} \supseteq M^\star$ is the convex hull of the tip of $M(0)$, P_{n+1}, of the $f_{i,n+1}$ and of the $f_{j,n+1}$ and includes points of reswitching that are not semi-positive, because there is a transgression at each vertex $j, j = m+1, \ldots, n$. The transgression drawn is weak, but the same result and almost the same diagram result, if it is strong. The reswitching body $M^\star$ contains only reswitch-points that are non-negative, hence it excludes the tails extending from the basis B of $M(0)$ to the $\bar{\mathbf{f}}_j$. But the existence of the tails implies that part of the basis B between F and $\mathbf{f}_j$ belongs to $M^\star$, hence also to the convex hull $M^{\star\star}$ of $M^\star$. The basis B is spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_n$. The borders E and F result from the intersections of the simplices spanned by $\bar{\mathbf{f}}_1, \ldots, \bar{\mathbf{f}}_n$ and $\bar{\mathbf{f}}_1^\star, \ldots, \bar{\mathbf{f}}_m^\star, \bar{\mathbf{f}}_{m+1}, \ldots, \bar{\mathbf{f}}_n$ with B respectively, where $\bar{\mathbf{f}}_i^\star$ stand for the minima of $f_{i,n+1}; i = 1, \ldots, m$.

The essential condition of *Theorem 3*, $\mu(D_1) \leq \mu(D_2)$, will be fulfilled only if m/n is close to one; moreover, we must have $(\mathbf{a}_1, l_1) > 0$. So this is only a possibility. A more general construction results, if we let S shift towards the bottom so that the space corresponding to D_2 in Figure 10 expands and that corresponding to D_1 contracts. The vertices of S on $h_{i,n+1}$ are represented by vectors $\tilde{\mathbf{f}}_i$ that originate in P_{n+1} and fulfil $\|\bar{\mathbf{f}}_i\| \leq \|\tilde{\mathbf{f}}_i\| \leq \|\mathbf{f}_i\|$. If continuous paths are prescribed for the movement of the $\tilde{\mathbf{f}}_i$ from $\bar{\mathbf{f}}_i$ to $\mathbf{f}_i$ – there is some freedom in choosing them – intermediate values $\tilde{\mathbf{f}}_i^\star$ will be found, for which what is now $\tilde{D}_1$ in Figure 11 is just accommodated in $\tilde{D}_2$ (if such an expansion is at all necessary, because it may be, as in Figure 10, that the mass of D_1 can be accommodated in D_2). The schematic diagram looks as if this could easily be done, because the area of D_1 ($\tilde{D}_1$) in Figure 10 (11) looks smaller than the area of D_2 ($\tilde{D}_2$), but these areas stand for partial volumina of simplices of dimension n, and there is more volume in the bottom near the basis than above it. Hence it is possible that all $\tilde{\mathbf{f}}_i$ will tend to $\mathbf{f}_i$, and if they do, the probability of reswitching is not zero, but one! Whether systems $(\mathbf{A}, \mathbf{l})$ exist that generate this kind of behaviour is not known. I doubt it, I certainly have never seen one, but, until a proof of impossibility is found, the formal possibility has to be taken into

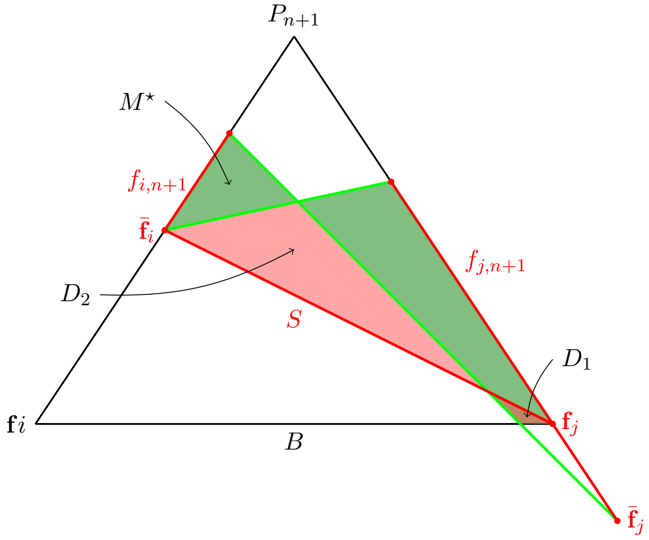


Figure 10: Simplex S connects the maxima $\bar{\mathbf{f}}_i$ of the $f_{i,n+1}$ of the empty vertices with the vertices $\mathbf{f}_j$ with transgressions. The part of $M^\star$ between S and basis B is transferred into the cavity between concave $M^\star$ and S.

account.

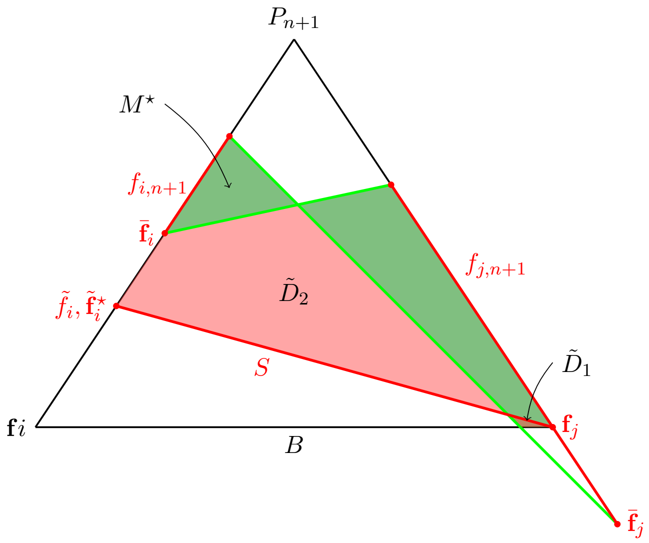


Figure 11: The volume $\tilde{D}_1$ of $M^\star$ enclosed between S and B is equal to the volume of $M(0)$ enclosed between S and the lower boundary of $M^\star$, thanks to the choice of the vertices $\tilde{f}_i^\star$ of S.

Theorem 4. *If, as n tends to infinity, a number $n^\star$ of $\tilde{\mathbf{f}}_i^\star$ with $\|\tilde{\mathbf{f}}_i^\star\| < \|\mathbf{f}_i\|$ can be found, which tends to infinity with n, the probability of reswitching tends to zero.*

Proof. If $M^{\star\star\star}$ is defined as the convex hull of P_{n+1} and S in $M(0)$, the conclusion follows as in the proof of *Theorem 3*. By the same consideration we get the

Corollary. *If all but a finite number of $\tilde{f}_i^\star$, say $i = 1, \ldots, k$, tend to $\mathbf{f}_i$, the probability of reswitching will tend to $\gamma_1 \cdot \ldots \cdot \gamma_k$, where $\gamma_i = \|\tilde{\mathbf{f}}_i^\star\|/\|\mathbf{f}_i\|$.*

Before discussing this result, we complement it with *Theorem 5*, for the proof of which we use another method to estimate the volume of $M^\star$ relative to $M(0)$: we use cones. Figure 12 shows that, after some rearrangement of the masses, we can represent the volume of $M^\star$ as contained in the volumes of two cones. Figure 8 illustrates how $M^{\star\star}$ extends to the basis of $M(0)$, if transgressions are involved. In order to be able to use the part of $M^\star$ pertaining to the basis as a measure of the extent of the transgressions, we make the simplifying assumption that the boundaries of $M^\star$ can be represented by two $(n-1)$-simplices (hence the boundaries are not only drawn, but actually assumed as flat), which connect the minima of $f_{i,n+1}$ with the maxima of $f_{j,n+1}$, and vice versa the maxima of $f_{i,n+1}$ with the minima of $f_{j,n+1}$, so that each of the two simplices, d_1 and d_2, are spanned by n points (see Figure 12), and $M^\star$ is now assumed to lie between them. The basis B, spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_n$, will be bisected by d_1, which runs across B; the separating set $b = d_1 \cap B$ is of dimension $n-2$. Let d_1 rotate around b until it contains the tip of $M(0)$, P_{n+1}, and denote this rotated simplex by d. This d then bisects not only B, into B_1 with vertices $\mathbf{f}_1, \ldots, \mathbf{f}_m$ and B_2 with vertices $\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n$, but the entire $M(0)$. Consider a hyperplane H of dimension $n-1$ parallel to and above B at distance δ to B. It cuts $\mathbf{f}_1, \ldots, \mathbf{f}_m$ and d; this domain of it is denoted by B_1'. B_1' will cut d_1 and, if δ is large enough, also d_2. Choose δ so that the volume V_1 below B_1' and between the boundary of $M(0)$ and d_2 on the one hand and the volume V_2 below B_1' and between d_1 and d on the other, taken together, are equal to the volume V_3 above B_1' and below d_1 and d_2. Since $V_3 = V_1 + V_2$, the two cones, the first formed by B_2 and P_{n+1}, bordered by d and the sides between $\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n$, of height $h = \hat{p}_1(0)$, and the second formed by B_1' and P_{n+1}, bordered by d and the sides between $\mathbf{f}_1, \ldots, \mathbf{f}_m$, of height $h - \delta$, will together contain the mass of $M^\star$ and the empty top of $M(0)$. Note that $\delta > 0$, because V_2 has only a set of dimension $n-2$ in common with B; this is the main difference in comparison with the construction of *Theorem 4* and Figure 11.

Now $\mu(B_1')/\mu(B)$ tends to zero with $n \to \infty$, because $\mu(B_1')/\mu(B_1)$ tends to zero. This follows, with $\beta = (h-\delta)/h$, from $\mu(B_1') = \beta^{n-1}\mu(B_1)$ as in the proof of *Theorem 1*. The volume of a cone of dimension n with basis $B^\star$ and height $h^\star$ is equal to the $(n-1)$-dimensional measure $\mu(B^\star)$, multiplied by $1/(n-1)!$ and by $h^\star$. $M(0)$ also is a cone. Hence the probability of reswitching is under the stated assumptions smaller or equal to

$$\frac{\dfrac{h-\delta}{(n-1)!}\mu(B_1') + \dfrac{h}{(n-1)!}\mu(B_2)}{\dfrac{h}{(n-1)!}[\mu(B_1) + \mu(B_2)]} = \frac{(1 - \dfrac{\delta}{h})\dfrac{\mu(B_1')}{\mu(B_1)} + \dfrac{\mu(B_2)}{\mu(B_1)}}{1 + \dfrac{\mu(B_2)}{\mu(B_1)}} \xrightarrow[n\to\infty]{} \frac{\mu(B_2)}{\mu(B)},$$

since $\mu(B_1')/\mu(B) \xrightarrow[n\to\infty]{} 0$ and $\delta > 0$. The volume of the top also tends to zero. Therefore we can state:

Theorem 5. *Given the simplifying assumptions in the text, summarized in the legend to Figure 12, the probability of reswitching tends to* $\mu(B_2)/\mu(B)$.

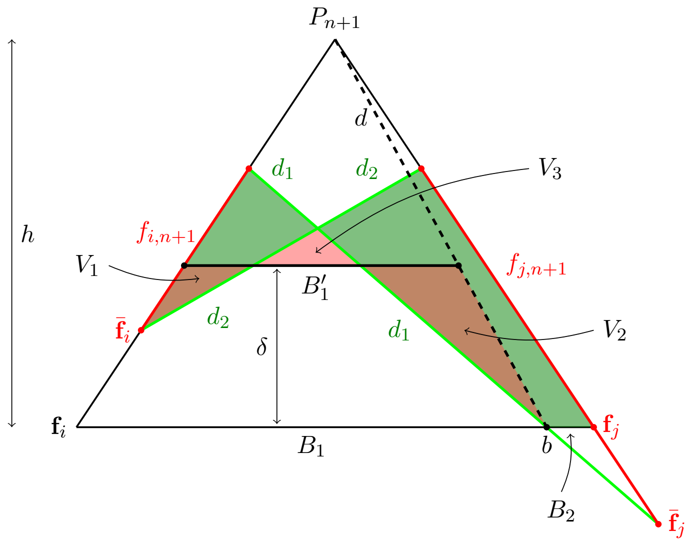


Figure 12: The assumptions are as for *Theorem 1*, with the additional assumption that the boundaries of $M^\star$ are flat, the x_i along $f_{i,n+1}; i = 1,\ldots,m$; are monotonically falling or do at any rate not rise beyond $z_i(0)$ and the transgressions are weak. B_2 borders $M^\star$ on basis $B = B_1 \cup B_2$. B_1', parallel to B_1, is chosen so that the volumina $V_1 + V_2$ equal V_3.

Our assumptions for *Theorem 5* are more restrictive than those for *Theorem 4* in particular, because $M^\star$ is here assumed to be bordered by flat surfaces. But we now used the star-shaped character of the reswitching body. *Theorem 5* and the *Corollary* to *Theorem 4* confirm that the probability of reswitching goes to zero for $n \to \infty$ only under assumptions, as had been emphasized in Schefold (2016), but they are now more detailed and explicit. The probability $\gamma_1 \cdot \ldots \cdot \gamma_k$ of the *Corollary* and the corresponding expression $\mu(B_2)/\mu(B)$ of *Theorem 5* will be small, and will tend to zero, if the number of transgressions diminishes relatively as n increases or if m/n increases. I regard this as plausible because the random character of technological systems becomes visible only if they are large. The Goldberg-Neumann theorem points in this direction, but the debate about this matter is not closed. Meanwhile, we have empirical evidence that transgressions are, relatively, not frequent. Taking this as an assumption, we have explained why reswitching is rare.

Note: On non-basics and new commodities
As indicated in the commentary to *Theorem 2*, non-basics can be treated as limit cases with $(\mathbf{a}_1, l_1)$ on the boundary of $M(0)$. There is only little interest in the matter in this paper, since we are here concerned with the foundations of capital theory, and capital goods are typically basic. The fact that non-basics played a significant role in the debate, explicitly in Sraffa, implicitly before him (e.g. in the exchanges between Irving Fisher and Eugen von Böhm-Bawerk) has to be explained by the limitations of the mathematical techniques available to the authors and by the intuitive appeal that the discussion of non-basics does indeed have, in particular for beginners.

We only consider the simplest case: there is one basic commodity and the non-basics are consumption goods. Hence $(\mathbf{a}_1, l_1)$ is on $h_{1,n+1}$ and f_{14} in Figure 8 shrinks to a point (assuming $n = 3$). One finds in this case, with

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}, \mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix},$$

$$\hat{p}_1(r) = \frac{l_1}{1-(1+r)a_{11}}, \hat{p}_2(r) = l_2 + \frac{(1+r)a_{21}l_1}{1-(1+r)a_{11}}, \hat{p}_3(r) = l_3 + \frac{(1+r)a_{31}l_1}{1-(1+r)a_{11}},$$

$$z_1(r) = \frac{1}{1+r}, z_i(r) = \frac{l_1}{(1+r)[l_i + (1+r)(a_{i1}l_1 - a_{11}l_i)]}; i = 2, 3; z_4 = \hat{p}_1(r).$$

The characteristic equation has two eigenvalues equal to zero, and zero is a double root of the characteristic equation. The system is therefore not regular; Bródy's hypothesis is fulfilled. Obviously $a_{11} = 1/(1+R)$, hence standard prices $p_i = (R-r)\hat{p}_i/R$ are linear

$$p_1 = \frac{1+R}{R}l_1, p_i = \frac{R-r}{R}l_i + \frac{1+R}{R}(1+r)a_{i1}l_1; i = 2, 3;$$

but $z_i(r)$ is not necessarily monotonically falling. It will fall, if $a_{i1}l_1 > a_{11}l_i$ or $a_{i1}/l_i > a_{11}/l_1$, that is, if sector one is less capital-intensive, while transgressions become possible in the converse case, though with further restrictions on the range of r, which we cannot discuss here fore reasons of space. Of course, one gets, using (10), that $x_1(r) = a_{11}$. The star-equation, (ix) of the *Lemma*, with $a_{12} = a_{13} = 0$, reduces to

$$a_{11}/z_1(r) + l_1/z_{n+1}(r) = 1,$$

showing the opposed movements of z_1 and z_{n+1}. These findings confirm that systems with non-basics can, with regard to the questions discussed in this paper, be viewed as limit cases and are useful for the intuition.

Non-basics can also be used to construct a counterexample to the theorems, which we have proved, because it violates the assumption that the number of transgressions grows less with n than the number of empty vertices. It is so intuitive that it suffices to describe it in verbal terms. The reader will be able to represent it formally, if that seems necessary.

The idea is to add to a given basic system (the corn model is a possibility) pairs of non-basics of the type of Sraffa's wine and oak-chest example. One year old wine is produced by means of labour alone and becomes successively, with a zero labour input, two years old, three years old and, eventually, eight years old wine, which is the consumption good. Similarly for the oak-chest, where a small amount of labour is used to plant the tree. The new tree becomes, with zero labour inputs, successively older until it is made into the chest by means of a considerable labour input. Such an example was first used by Irving Fisher, long before Sraffa, and by Samuelson after him. With appropriate values for the positive labour inputs, one gets reswitching, in that wine and oak chest will be of equal price at two different rates of profit over a certain range, and one can express this price in terms of the standard commodity of the basics, if one wishes, as Sraffa does, although wine and oak-chest are not inputs to the production of other commodities; they are consumption goods (the model is Austrian). An ever-increasing number of pairs of such non-basics can be added to the given and unvariable basic system. What this means for the probability of reswitching is not clear before one defines the ranges of parameter values within the set of potential techniques. Various definitions can be thought of. Several authors have come to the conclusion that the probability for reswitching is small, but positive in Austrian models (see Section 1) – the three positive labour inputs to the production of wine and oak-chest must vary in restricted domains to produce the desired effect – but it becomes clear that the construction of reswitching examples is easy with non-basics. Of course, several of the assumptions used in Section 6 do not hold, so we get to a different conclusion. If the probability for reswitching is ϵ for a given pair and if we have m such pairs with the same ranges for the parameter values, the probability that reswitching occurs at least once is $1-(1-\epsilon)^m$ and tends to certainty with $m \to \infty$, even if ϵ is small.[4] The construction is illustrative but the model with the indefinitely large number of pairs is special; it does not take anything away from what we found in this paper, since capital goods are essentially basics and the construction cannot be extended to basic commodities without modifications – see Section 7.

Using this rudimentary theory of non-basics, we can discuss the introduction of new goods in the system. Petri (2011) claimed that the probability of reswitching increases substantially, if a switch of technique is associated with the production and use of new goods, but he refers to the Samuelson or corn-tractor model, which we criticised in Section 1, because it does not show how one gets from the system using horses to the system with tractors.

[4]We can vary Sraffa's example to create a case with $\epsilon = 1$. The oak-chest is produced with one unit of labour 25 years ago and 19 units of labour in the present. The wine is produced by λ units of labour. In Sraffa's case, $\lambda = 20$, and the price of the oak-chest in labour commanded is $\hat{p}_c = 19 + (1+r)^{25}$ and that of wine is $\hat{p}_w = 20(1+r)^8$. We have $\hat{p}_w = \hat{p}_c$ for $r = 0$ and $r \approx 17\%$, hence we would have a kind of reswitching, if these two methods produced the same commodity. We estimate the probability of this reswitching by assuming that one method, here the oak-chest method, is given, and the possible techniques – here the wine technique – have a switch in common with the oak-chest technique. From $\hat{p}_c = 19 + (1+r)^{25} = \hat{p}_w = \lambda(1+r)^8$, one gets $\lambda = (19 + (1+r)^{25})/(1+r)^8$. If one plots this curve, one finds that it is in the relevant positive range similar to a parabola with its minimum of $\lambda \approx 13.86$ at $r \approx 11\%$. Hence, if the labour input is below 13.86, there is no solution (but such a labour input would be economically admissible), while, if $\lambda > 13.86$, there will be two solutions. Hence, if there is one switch, such as for $\lambda = 20$, there is also another, and reswitching is certain and $\epsilon = 1$.

The emergence or disappearance of commodities in the system has been discussed in the context of joint production. If wages are lowered, it may pay to use an old lorry of age T one year longer. A new commodity appears in the system in the form of a lorry of age $T + 1$ (previously, the lorry had been discarded), together with the process of using it, producing the transport of, say, wheat. This is the application of a broader principle. The single production of commodities is, as a rule, associated with the production of by-products, such as wastes, which fetch no price because they are not used. As soon as a use for one of the wastes has been discovered, it becomes a commodity and enters the system (Schefold 1997, pp. 197-239). New commodities can thus be discussed within the Sraffa framework, but we want here to avoid joint production, because of the complications this would involve.

Another way to introduce new goods is to assume that they emerge as non-basics, because their use is first tested only by some firms; the wage curve, say w_1, then results from the prices of basics. The good, e.g. a loom, becomes a truly basic commodity, when it is generally used in the textile industry. We then get a new system, in which the process producing the loom is retained, and the loom is used in the textile industry with a somewhat changed composition of inputs.

We thus have an introduction of the loom in two steps. First, there is the old system without the loom; the wage curve is denoted by w_0. Then, part of the surplus is used to produce the loom (progress results from new combinations, as Schumpeter said), and we have a new wage curve, which we denote by w_1. Then the loom is generally used and we get wage curve w_2 in terms of the same commodities that represented the numéraire in w_1.

To analyse the possibilities for isolated reswitching, we first compare w_0 and w_1, but these wage curves are identical. Then we compare w_1 and w_2. Since we assume that the loom is produced by part of the surplus (the new combination), there is a non-basic, but we do not have an Austrian model. We therefore remain within the framework which has been developed in the paper. Contrary to Petri (2011), we need not change our conclusion because of new goods.[5] The systems represented by w_0 and w_1 differ only in a non-basic process, which does not affect the wage curve. The systems represented by w_1 and w_2 differ only in one process, and the probability for an isolated reswitch can be analysed.

7 The probability of reverse capital deepening and final considerations

An important application of Theorems 1 – 5 concerns the critique of Samuelson's production function on the basis not of reswitching, but of estimating the number of wage curves on the envelope of the wage curves of many techniques. It is asserted in Kersting and Schefold (2021) that the number of wage curves on the envelope, that is, the number of techniques that are efficient and may come into use in consequence of shifts in distribution, is much smaller than Samuelson's construction suggested. The proof uses assumptions about probability distributions of maximum rates of profit and maximum

[5] See also Petri (2021, 2022) and Schefold (2022b, 2022c).

wage rates. This means a new turn in the critique of capital. The efficient techniques are few so that there is not much room for substitution (Schefold 2021a). According to the result that can most easily be reached, mentioned in Section 1, the number of efficient techniques to be expected is at most $\ln s$, if s is the number of available techniques. The proof of this formula in Schefold (2013b) uses the assumption that wage curves are straight lines, because that is what Samuelson assumed. However, the crucial assumption is not linearity or quasi-linearity, but that the wage curves that reach the envelope through a first switch do not exhibit a second (no double switching, see Kersting and Schefold (2021, p. 523)). Now we have proved that second switches are unlikely. By implication, we have proved – provided the corresponding randomness assumptions hold – that the expected number of wage curves on the envelope is at most $\ln s$, even if wage curves are not straight. It is ironic that the new turn in capital theory profits from the absence of reswitching, while reswitching and reverse capital deepening had been the main and most conspicuous arguments in the first debate.

We now turn to the probability of reverse capital deepening. To define the probability of isolated reswitching, we started from a given system – one method for each commodity, and one alternative method in the first process. An admissible alternative method could in principle be any semi-positive vector $(\mathbf{a}_0, l_0)$, superior, inferior or equiprofitable, relative to $(\mathbf{a}_1, l_1)$, but the set of possible techniques is that of the equiprofitable techniques at *one* given rate of profit r_1 and the set of favourable techniques is the subset of the possible techniques which are equiprofitable at *some* other rate of profit r_2. The set of admissible alternatives is the semi-positive quadrant $\mathbb{R}_+^{n+1} - \{0\}$ of dimension $n + 1$, the possible methods are in the n-dimensional simplex $M(r_1)$, the favourable methods are $M^\star(r_1)$, the union of the intersections of $M(r_1)$ with $M(r_2)$, $0 \leq r_2 \leq R, r_2 \neq r_1$. $M^\star$ is of dimension $n-1$, if the labour theory of value holds, but a star-shaped region in $M(r_1)$, if the system is regular so that relative prices change with the rate of profit.

The reader, who has patiently read the paper up to this last section, will have understood this, but perhaps without noticing the following ambiguity in this definition: If there are two techniques with wage curves w_1 and w_2 that intersect twice, so that we have reswitching (which is, in the absence of other techniques, at the same time a case of isolated reswitching, of systemic reswitching and of reverse capital deepening) and if we now ask how probable this constellation is, we find that we get different results depending on which system is regarded as the first and which switch is regarded as given. The ambiguity did not come up earlier, because of the way in which we posed the question: We asked for the probability *ex ante*, if a new method was introduced and produced a switch at some r_1 – would there result another switch at r_2, with what probability? *Ex post*, one may not be interested any more in the sequence of the events; then, one will get different results according to the sequence one assumes. The difference will usually be small. Its existence is not really a surprise, since the probability of isolated reswitching is defined relative to a given technique. Of special interest will therefore be those results, which do not depend on the assumption of the sequence, in particular that the probability of isolated reswitching tends to zero with the number of sectors.

Reverse capital deepening occurs on the envelope of the wage curves with large systems and many techniques. It will now matter how many switch-points and wage curves there are on the envelope according to the results of Kersting and Schefold (2021).

We consider any switch-point on the envelope engendered by two wage curves, denoted by w_1 and w_2, and we assume that reverse capital deepening is associated with it. Since we are on the envelope, the two techniques differ only in one method of production in one sector, say the first. We know or choose the 'prior' method $(\mathbf{a}_1, l_1)$, the 'secondary' is $(\mathbf{a}_0, l_0)$. Either this switch-point in r_1 is itself a case of reverse capital deepening (case A) or w_1 and w_2 intersect elsewhere at some r_2 and reverse capital deepening occurs at r_2 (case B).

Consider case B first. Since there is no capital reversal at r_1, we must have $r_2 > r_1$ and the switch at r_2 is on the envelope (otherwise it is no capital reversal). Disregarding further switches, because they would be unlikely, both switches are therefore on the envelope; case B implies that we have systemic reswitching. But it is unlikely that neither switch-point is dominated, if there are many wage curves on the envelope, and so we exclude case B, as being of a low probability which may be disregarded.

This may not be said of case A, since the other switch-point must be at some $r_2 < r_1$ and may be (it probably will be) below the envelope. If we abstract from all other techniques except those used for w_1 and w_2, we then have, by the construction, isolated reswitching with a positive probability that tends to zero with an increasing number of sectors. This probability will also (and strongly) depend on r_1, for the switch-point r_2 must be between zero and r_1, and there will be the more room for such a switch-point, the closer r_1 is to R. This effect is very visible in the results of the empirical work by D'Ippolito (1989, p. 196). So we have countervailing tendencies. The probability that a switch-point represents reverse capital deepening is smaller than that of isolated reswitching, because we exclude case B, but it approximates that of isolated reswitching as the switch-point gets close to the maximum rate of profit of the first technique. It diminishes with an increasing number of sectors. Nonetheless, the probability that some switch-points on the envelope of a system with many techniques exhibit reverse capital deepening obviously increases with the number of switch-points. What prevails?

Our heuristic methods do not allow to derive a definite answer. The complication increases because the number of switch-points and the number of sectors may be connected. I propose two illustrative outcomes, tentatively using audacious assumptions.

(i) The probability π_n that any switch in an n-sector model with many techniques is a case of reverse capital deepening can only be understood as a rough average for classes of systems with not too many transgressions. We have convinced ourselves in this section that π_n must be smaller than the probability for isolated reswitching. An exact general rule can hardly exist, but we suppose $\pi_n = \frac{\alpha}{n}$, where α would be of the order of magnitude of 0.3 for the following reasons: If $n = 30$, the results of Han and Schefold (2006) and of Zambelli (2018), as interpreted by Kalb (2022), indicate a π_n for systems with about 30 sectors of about 1%. That implies $\alpha = 0.3$. If the formula is applied to a three sector model, one obtains $\pi_3 = 0.3/3 = 10\%$, which is high, but not altogether implausible. We denote the number of switch-points by ω and assume $\omega = \ln s$, according to Schefold (2013b), where s is the total number of techniques. Suppose that s is given by the methods employed in m countries with n industries, hence $s = m^n$. We keep m constant as n increases. The probability $\hat{\pi}$ that at least one case of reverse capital deepening appears then is, with $m = 10$

and $\ln m \approx 2.3$,

$$\hat{\pi} = 1 - (1-\pi_n)^{\omega} = 1 - \left(1 - \frac{0.3}{n}\right)^{n \ln m} \xrightarrow[n\to\infty]{} 1 - e^{-0.3\cdot 2.3} \approx 1 - e^{-0.69} \approx 0.5.$$

That the probability of finding at least one case of reverse capital deepening should be so low where there are so many techniques is surprising but a similar consideration confirms the result.[6] Further research is required to test the robustness of the result, more theory is needed to improve on the assumptions of π_n as an average, inversely proportional to n. The number 0.5 for $\hat{\pi}$ is therefore only a rough estimate, but it conveys an important message: the probability of reverse capital deepening occurring in a large system is less than one, despite a tendentially infinite number of switch-points. Most people would probably expect that this probability $\hat{\pi}$ tends to one as the number of switch-points ω tends to infinity. Two reasons are jointly needed to explain why the opposite happens: clearly, π_n must tend to zero, according to the main result of the paper: the probability of isolated reswitching tends to zero. For otherwise $\pi_n \to \bar{\pi}, 0 < \bar{\pi} < 1$, and $(1-\bar{\pi})^{\omega} \xrightarrow[\omega\to\infty]{} 0$ and $\hat{\pi} \to 1$. And it matters how the number of switch-points goes to infinity with the number of sectors. It is likely that the number of techniques increases exponentially with n. We used the assumptions m^n – in each of n industries, one of the m methods employed in m countries can be used. Suppose that only a fraction ζ, $0 < \zeta << 1$, of these techniques matters economically, because it is transferable to the country under consideration. Suppose now that, contrary to Kersting and Schefold (2021), each additional transferable technique engenders one additional switch-point as in Samuelson's surrogate production function, if each additional transferable linear wage curve gets on the envelope, because the maximum rates of profit are in the inverse order of the maximum wage rates. Then $\omega = \zeta m^n$ and

$$\hat{\pi} = \lim_{n\to\infty} 1 - \left(1 - \frac{\alpha}{n}\right)^{\zeta(m^n)} = 1 - \lim_{n\to\infty}\left(1 - \frac{\alpha}{n}\right)^{n(\zeta/n)(m^n)} = 1 - \lim_{n\to\infty}\left(e^{-\alpha\zeta}\right)^{(m^n)/n}.$$

Since $0 < \alpha\zeta, e^{-\alpha\zeta} < 1$ and since $(m^n)/n \xrightarrow[n\to\infty]{} \infty$ with $m > 1, \hat{\pi} \xrightarrow[n\to\infty]{} 1$. Reverse capital deepening becomes certain despite $\pi_n \to 0$, but only if the number of switch-points increases exponentially with n. The density of the switch-points with RCD, $\psi/\omega = \pi_n$, would tend to zero, however.

In other words: the tendency of the probability of isolated reswitching to go to zero implies that, in large systems, not even one case of reverse capital deepening would be certain to be observed, as long as the number of switch-points increases not exponentially with the number of sectors, but only logarithmically – or even less fast.

[6] There were 496 envelopes in the empirical investigation by Han and Schefold (2006), each representing the envelope resulting from the combinations of 33 methods in $n = 33$ industries in two countries, so that there were 2^{33} techniques underlying each envelope. Assuming uniform distributions in order to apply Kersting's theorem, we now have $\omega = \frac{2}{3}\ln s$. The expected number of cases of reverse capital deepening ψ then is $\psi = \pi_n\omega = \frac{0.3}{33}\cdot\frac{2}{3}\ln s = 0.3\cdot\frac{2}{3}\ln 2 = 0.14$. We get for the expected number of cases of RCD on all envelopes $\bar{\psi}$: $\bar{\psi} = 496\psi \approx 69$. Han and Schefold (2006) found 60 cases of RCD on 4389 switch-points on all envelopes taken together.

ii) In fact, this is confirmed, if further results from Kersting and Schefold (2021) are taken into account. Then, the number of switch-points to be expected in the relevant range of the envelope remains finite. In the case of uniform distributions of maximum wage rates and uniform rates of profit, there may not even be two switch-points in the relevant range of the rate of profit (Kersting's theorem). If the two distributions of maximum wage rates and maximum rates of profit are normal with a moderate correlation, numerical experiments with large numbers of techniques also resulted in a small number of wage curves appearing on the envelope. The individual wage curves were assumed as linear, but they could also be interpreted as short cuts of non-linear wage curves. The conclusion in the paper was that there is not much room for substitution between capital and labour. The conclusion which follows here is that there is not much room for reswitching and reverse capital deepening either.

To summarise, we get a complementarity: Either the number of switch-points on the envelope increases less than exponentially with the number of sectors. Then, it is not certain that even one case of reverse capital deepening exists in a large system, but the possibilities of substitution are limited. Or the number of switch-points increases exponentially. This seems to be fundamental for Samuelson's surrogate production function. Then, reverse capital deepening is almost certain to exist.[7] Further research will have to show how this conclusion needs to be modified, if the analysis is improved and if, in particular, the simplifying assumption of an inverse proportionality between the probability for isolated reswitching and the number of sectors is overcome.

[7]The complementarity does not imply that there is a strict causal link between zero substitution and the absence of reswitching. It is formally possible – though unlikely – that there are many near-linear wage curves without reverse capital deepening (the Samuelson case). Such a constellation is, as it were, doubly improbable, because it is unlikely that there are many wage curves on the envelope in the relevant range (as is shown in Kersting and Schefold 2021) *and* it is improbable (though possible) that there is then no RCD at all. Conversely, it is possible, though improbable, that there are few wage curves on the envelope and yet there is reswitching. The complementarity thesis therefore only states what seems to be a rule, capable of exceptions, given the present state of our knowledge.

References

Bienenfeld, M. (1988). Regularity in price changes as an effect of changes in distribution. *Cambridge Journal of Economics*, 12(2): 247-255.

D'Ippolito, G. (1987). Probabilità di Perverso Compartamento del Capitale al Variare del Saggio dei Profitti: Il Modello Embrionale a Due Settori. *Note Economiche*, 2: 5-37.

D'Ippolito, G. (1989). Delimitazione dell'Area dei Casi di Comportamento Perverso del Capitale in u Punto di Mutamento della Tecnica. In *Aspetti Controversi della Teoria del Valore*. Ed. by Luigi Pasinetti, Bologna: Il Mulino, 191-197.

Eltis, W. A. (1973). Some Problems in Capital Theory. In *Growth and Distribution*, Chapter 5, London: The Macmillan Press Ltd., 93-125.

Ferrer-Hernández, J., and Torres-González, L. D. (2022). Some recent developments on the explanation of the empirical relationship between prices and distribution. *Centro Sraffa Working Papers*, No. 54.

Fisher, I. (1907). *The Rate of Interest.* New York: Macmillan.

Goldberg, G. and Neumann, M. (2003). Distribution of subdominant eigenvalues of matrices with random rows. *SIAM Journal on Matrix Analysis and Applications*, 24(3): 747-761.

Han, Z. and Schefold, B. (2006). An empirical investigation of paradoxes: reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics*, 30(5): 737-765.

Hicks, J. R. (1973). *Capital and Time: A Neo-Austrian Theory.* Oxford: Oxford University Press.

Iliadi, F., Mariolis, T., Soklis, G., and Tsoulfidis, L. (2014). Bienenfeld's approximation of production prices and eigenvalue distribution: further evidence from five European economies. *Contributions to Political Economy*, 33(1): 35-54.

Kalb, J. (2021). Visualising the probability of reswitching. *Unpublished Working Paper (in German).*

Kalb, J. (2022). Reverse capital deepening remains rare. A comment on Zambelli's 'The aggregate production function is NOT neoclassical'. *Unpublished Working Paper.*

Kersting, G., and Schefold, B. (2021). Best techniques leave little room for substitution. A new critique of the production function. *Structural Change and Economic Dynamics*, 58: 509-533.

Krelle, W. (1977). Basic facts in capital theory: some lessons from the controversy in capital theory. *Revue d'économie politique*, 87(2): 282-329.

Laing, N. (1991). The likelihood of capital-reversing and double-switching. *Bulletin of Economic Research*, 43(2): 179-188.

Levhari, D. (1965). A nonsubstitution theorem and switching of techniques. *The Quarterly Journal of Economics*, 79(1): 98-105.

Mainwaring, L. and Steedman, I. (2000). On the probability of re-switching and capital reversing in a two-sector Sraffian model. In *Critical Essays on Piero Sraffa's Legacy in Economics*. Ed. by Heinz D. Kurz. Cambridge: Cambridge University Press, 323-354.

Mariolis, T., and Tsoulfidis, L. (2009). Decomposing the changes in production prices into"capital-intensity" and"price" effects: Theory and evidence from the Chinese economy. *Contributions to Political Economy*, 28(1): 1-22.

Marzi, G. (1994). Vertically integrated sectors and the empirics of structural change. *Structural Change and Economic Dynamics*, 5(1): 155-175.

Ochoa, E. M. (1989). Values, prices, and wage-profit curves in the US economy. *Cambridge Journal of Economics*, 13(3): 413-429.

Petri, F. (2011). On the likelihood and relevance of reswitching and reverse capital deepening. In *Keynes, Sraffa and the Criticism of Neoclassical Theory: Essays in Honour of Heinz Kurz*. Ed. by Neri Salvadori and Christian Gehrke. London: Routledge, 380-418.

Petri, F. (2021). What remains of the Cambridge critique? On Professor Schefold's theses. *Centro Sraffa Working Papers*, No. 50.

Petri, F. (2022). What remains of the Cambridge critique? On Professor Schefold's theses. *Contributions to Political Economy*, 41, 83-115.

Petrović, P. (1991). Shape of a wage-profit curve, some methodology and empirical evidence. *Metroeconomica*, 42(2): 93-112.

Ricardo, D. (1951). *The Works and Correspondence of David Ricardo*. Ed. by Piero Sraffa with the Collaboration of M.H. Dobb. University Press for the Royal Economic Society.

Samuelson, P. A. (1962). Parable and realism in capital theory: the surrogate production function. *The Review of Economic Studies*, 29(3): 193-206.

Samuelson, P. A. (1966). A summing up. *The Quarterly Journal of Economics*, 80(4): 568-583.

Schefold, B. (1971). Piero Sraffas Theorie der Kuppelproduktion, des fixen Kapitals und der Rente (Mr. Sraffa on Joint Production). Private Print. Online as: `https://www.wiwi.uni-frankfurt.de/fileadmin/user_upload/dateien_abteilungen/abt_ewf/Economic_Theory/publications/Theorie_der_Kuppelproduktion_Bertram_Schefold_1971.pdf`.

Schefold, B. (1976). Relative prices as a function of the rate of profit: a mathematical note. *Zeitschrift für Nationalökonomie/Journal of Economics*, (H. 1/2): 21-48.

Schefold, B. (1997). *Normal Prices, Technical Change and Accumulation.* London: MacmIllan.

Schefold, B. (2013a). Approximate surrogate production functions. *Cambridge Journal of Economics*, 37(5): 1161-1184.

Schefold, B. (2013b). Only a few techniques matter! On the number of curves on the wage frontier. In *Sraffa and the reconstruction of economic theory: Volume one.* Ed. by Enrico Sergio Levrero, Antonella Palumbo and Antonella Stirati. London: Macmillan, London, 46-69.

Schefold, B. (2016). Marx, the production function and the old neoclassical equilibrium: Workable under the same assumptions? With an appendix on the likelihood of reswitching and of Wicksell effects. Centro Sraffa Working Papers, No. 19.

Schefold, B. (2017). *Great Economic Thinkers from the Classicals to the Moderns.* New York: Routledge.

Schefold, B. (2021a). Transformations of the Cambridge critique. *The Indian Economic Journal*, 69(2): 241-254.

Schefold, B. (2021b). Sraffa's 'reduction' of the prices of basics. *Structural Change and Economic Dynamics*, 59: 575-581.

Schefold, B. (2022a). From reswitching to zero substitution: a surprising turn in the debate on capital theory. *Acta Oeconomica* (forthcoming).

Schefold, B. (2022b). What remains of the Cambridge critique? Potential conclusions and directions for further research following from recent investigations in capital theory. Centro di Ricerche e Documentazione" Piero Sraffa". *Centro Sraffa Working Papers*, No. 53.

Schefold, B. (2022c). What remains of the Cambridge critique? A new proposal. *Contributions to Political Economy*, 41: 116-135.

Shaikh, A. (1998). The empirical strength of the labour theory of value. In *Marxian economics: A reappraisal*. Ed. by Riccardo Bellofiore. London: Palgrave Macmillan, pp. 225-251.

Shaikh, A. (2016). *Capitalism: Competition, conflict, crises*. Oxford: Oxford University Press.

da Silva, E. A. (1991). The wage-profit curve in Brazil: an input-output model with fixed capital, 1975. *Review of Radical Political Economics*, 23(1-2): 104-110.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*. Cambridge: Cambridge University Press.

Tsoulfidis, L., and Maniatis, T. (2002). Values, prices of production and market prices: some more evidence from the Greek economy. *Cambridge Journal of Economics*, 26(3): 359-369.

Tsoulfidis, L., and Rieu, D. (2006). Labor values, prices of production, and wage-profit rate frontiers of the Korean economy. *Seoul Journal of Economics*, 19(3): 275-295.

Tsoulfidis, L. (2008). Price-value deviations: further evidence from input-output data of Japan. *International Review of Applied Economics*, 22(6): 707-724.

Woods, J. E. (1988). On switching of techniques in two-sector models. *Scottish Journal of Political Economy*, 35(1): 84-91.

Zambelli, S. (2018). The aggregate production function is NOT neoclassical. *Cambridge Journal of Economics*, 42(2): 383-426.