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Christian Bidard

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Christian Bidard[*†]

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Abstract

We propose to re-read Ricardo's theory of rent to which, we claim, the post-Sraffian literature is methodologically unfaithful. Ricardo's dynamic approach follows the transformations of a long-term equilibrium with demand. Sraffa adopted the same framework while substituting a value criterion for a physical criterion to determine the incoming marginal method, but he did not state the law of succession of methods explicitly. This prevented him to realize that his critique to Ricardo opens the door to all complications of capital theory, with the consequence that the Ricardian dynamics fail when a divergence appears between profitability and productivity. Contemporary studies have cast doubts on the validity of some of Ricardo's and Sraffa's over-optimistic conclusions, but the abandonment of the dynamic approach does not allow them to explain the ultimate reason of the phenomena they have pointed at. Ricardo's method has been recently rediscovered by mathematicians.

Keywords. Classical theory, land, rent, Ricardo, Sraffa

JEL classification. B12, B51, C61, D33

[*]University Paris-Ouest, EconomiX, 200 avenue de la République, F-92001 Nanterre. E-mail: christian.bidard@u-paris10.fr

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1 Introduction

The[1] publication of five essays by Malthus, Torrens, West and Ricardo in February 1815 constituted a decisive step in the development of the Classical theory of rent (details in Sraffa, 1954, p. 5). The question of lands and rent, which had been analysed by Smith (also by Anderson, 1777a and 1777b), was at the core of the struggle for economic and political power between landowners and the rising capitalist class, and the discussions on the corn laws at the Parliament triggered the economists' interest on rent and its policy implications. Ricardo's *Essay on Profits* is a reply to Malthus's two pamphlets (*Inquiry* and *Grounds*) in which Malthus developed protectionist arguments. By contrast, Ricardo used his theory as a plea in favour of free trade, which planes rents and improves the general rate of profit. He held a similar position in the *Principles* (1817), in which he made use of a more precise theory of value and described the process of extension and intensification of cultivation in more details.

We propose to return to the analytical side of Ricardo's construction and its legacy in the Classical tradition (see Pasinetti (2014) on that tradition). Ricardo adopted the labour theory of value for reproducible commodities and used the property that the marginal capital pays no rent to extend its field of application to agricultural products: the labour values of commodities are defined by the industrial methods and the marginal agricultural methods; these values once known, the conditions of production on intra-marginal lands (those which are already fully cultivated, as they are of a better quality than the marginal lands) determine the rents of those lands as differential costs. The price of corn being independent of its conditions of production, the owners of the best lands are in a position to demand a rent from their farmers. Sraffa's (*PCMC*, 1960) analysis constitutes the true line of descent of Ricardo's approach. Sraffa referred to prices of production instead of labour values and criticised some aspects of Ricardo's construction, but his methodology is faithful to Ricardo: he adopted the dynamic approach, which consists in following the transformations of a long-term equilibrium when demand in-

[1] A lighter version of the same beverage, with less references to Ricardo's and Sraffa's works, an abridged appendix and no added sugar, is sold by the EconomiX company as "The Ricardian rent two centuries after". Whatever version it is, this natural and biological product is recommended by the Food and Drug Administration to persons in general good health suffering from difficulties in understanding rent theory, especially to patients who show signs of allergy to equations.

creases, the dramatic moment being reached when a scarcity constraint is met on some land. Then some marginal method (outgoing marginal method) is replaced by another (incoming marginal method). We characterise that approach in Section 2 and identify its central questions as a search in two directions: (i) the law of succession of methods when demand changes; and, (ii) the reduction of the properties of a productive system with lands to those of a system without lands. We propose a re-reading of Ricardo and Sraffa in view of the answers they provide to these questions. Concerning the succession of methods, Sraffa determined the incoming marginal method by means of a value criterion instead of Ricardo's physical criterion, and this led him on the verge of an explicit statement of a general law, but he failed to do so. That law is stated in Section 3. In Section 4, we look at the interactions between the two aspects of Ricardo's and Sraffa's researches. It turns out that, for two independent reasons, the value criterion may prevent the reduction of a productive system with land to a single-product system, a failure illustrated by the violation of the trade-off property between wages and profits. Moreover, Sraffa did not notice that, once the value criterion is put forward, the critique he addressed to capital theory has its counterpart in the analysis of production with land, with the effect that the working of the dynamics themselves is not guaranteed (Section 5).

Sraffa's work on land initiated a field of researches which we identify as 'post-Sraffian'. Among other valuable results, we stress three significant conclusions: a general existence theorem (Salvadori, 1986), a necessary and sufficient criterion for uniqueness (Erreygers, 1990, 1995) and a number of results illustrating the general idea that the behaviour of productive systems with lands is more complex than suggested by Ricardo and Sraffa. For instance, there may exist several long-run equilibria sustaining a given level of final demand, while uniqueness is ensured for single-product systems without lands. Conclusions of that type cast doubt on the feasibility of Ricardo's programme, which was also adopted by Sraffa in its modernized version.

The post-Sraffian literature on lands is often involved and rarely hesitates to drown the reader under an impressive technical apparatus and the weight of equations. Our main critique, however, is that most of that literature follows a static approach, the question becoming that of the search of a cost-minimising system for a given level of demand. It is argued in Section 6 that the abandonment of the dynamic approach does not allow to explain the origin of some apparent paradoxes. In Section 7 we reinterpret the results relative to existence and uniqueness in terms of the dynamics. The final Section

8 shows Ricardo as an unexpected precursor of a contemporary mathematical tool.

Except in a few cases when interindustrial relationships really matter, as for the general statement of the law of succession or in Section 4.2, there is no harm in considering that the model we refer to is a corn model, with corn as the unique produced good (or the unique basic good) in the economy. Corn can be cultivated on different lands (theory of extensive rent) or on the same land by several methods (theory of intensive rent), or both. The hypothesis of a unique good suffices to understand the structure of rent theory and the difficulties it meets, and the simplicity of the retained framework aims at discarding the common opinion that rent theory would be a complex matter or that its main difficulties would start with the multiplicity of agricultural products.[2] Rent theory is first a question of method: the law of succession of methods is the Ariadne's thread of the whole construction.

2 Aim and methodology

For Ricardo (1817), the labour theory of value provides the tool for understanding the working of the forces at stake in a capitalist economy and, in particular, for explaining prices. That theory allowed him to state the trade-off property between wages and profits. However, it only applies to reproducible commodities and not, *a priori*, to land or commodities produced by means of lands. This is why, immediately after having introduced the notion of labour value, Ricardo examined the case of production with lands (*Principles*, Chapter 2). His extension of the labour theory of value to agricultural products is based on the analytical possibility of 'getting rid of rent' (letter to McCulloch, 13 June 1820) thanks to the property that 'the capital last employed pays no rent' (*Principles*, Chapter 2). As a consequence, the labour theory of value still applies to the industrial methods and the marginal agricultural methods. At prices determined by these methods, a land of a higher grade yields a rent equal to the differential cost of production with the marginal land. To sum up, the successive steps in the determination of an equilibrium are: the level of demand defines which lands are cultivated (or, in the case of intensive cultivation, which methods are used), the marginal

[2]Readers interested in general formalisation and proofs are invited to look at Appendix B. Apart that Appendix, the only requirements are some familiarity with Ricardo's theory and Sraffa's formalisation and basic mathematical knowledge.

methods define the prices, and a comparison with the marginal conditions determines the rents on intra-marginal lands. For a given level of demand, the trade-off property between profits and wages still holds and, in the face of an increasing demand and a given real wage, rents rise at the expense of profits.

Sraffa first studied prices of production for single-product systems. As production with lands involves joint production, the usual economic laws must be adapted: for instance, the absence of a positive standard basket is linked to the fact that lands are non-basic. However, as far as prices and distribution are concerned, Sraffa's aim parallels Ricardo's and, in the absence of any opposite hint, Sraffa seems to share the opinion that the results proved for single-product systems still apply. Even if he mentioned the multiplicity of agricultural products and lands as a potential source of complications, the only difficulty he pointed at is linked to the construction of a standard commodity, and Sraffa concluded that 'in the case of a single quality of land, the multiplicity of agricultural products would not give rise to any complications' (*PCMC*, Section 89). The approach developed in the present paper holds in a very general framework, including the cases of multiple lands, multiple agricultural products and joint production, but, except in incidental remarks, we shall retain the hypothesis of a unique agricultural good, because that simple framework suffices to understand the structure of rent theory and the difficulties it meets.

The most significant hint of Sraffa's agreement with Ricardo's global project is of a methodological nature. Two distinct approaches to the study of production with land can be conceived. The static approach consists in writing down a system of equalities and inequalities for a given level of demand (or, in Sraffa's words, for given requirements for use). Then a long-term position is defined as a solution to these equations, which involve both physical and value conditions which will be examined below in more details (Section 6 and Appendix B). Ricardo did take such conditions into account and, for instance, set the nullity of rent on partially cultivated lands. But the dynamic approach he privileged is different: it is based on the study of the transformations of equilibrium with demand. The basic property is that, most of the time, a slight change in final demand is met by a slight adaptation of activity levels with, on the physical side, no changes in the list of operated methods and cultivated lands and, on the value side, no changes in prices and rents. The adaptation of activity levels only concerns the methods already in use and consists in extending cultivation on a partially cultivated land or extending the use of a more productive method on a fully cultivated land. A limit to that adaptation is reached when a scarcity constraint is met. Then the price of corn jumps to a higher level and a new marginal method is introduced. The rents on cultivated lands rise suddenly with the price of corn. After that shock, a new equilibrium is found and another period of calm opens again, with a smooth adaptation to changing physical requirements.

Sraffa's adopting the same scheme in Chapter 11 of *PCMC* is all the more noteworthy that the dynamic approach he follows contradicts the explicit warning of the Preface according to which '[n]o changes in output and (at least in Parts 1 and 2), no changes in the proportions in which different means of production are used in an industry are considered'. As the adaptation of activity levels during calm periods sets no difficulties, the main point of the dynamic approach is the study of the phenomena which occur under critical circumstances. The phenomenon is striking because the 'spasmodic' (*PCMC,* Section 88) change of method it involves goes with a discontinuity in prices and rents. The dynamics, however, are not chaotic. First, when a scarcity constraint is met, there is no complete reorganisation of production, as the economy reacts by changing only one marginal method. Second, activity levels vary continuously with demand during calm periods and, we stress, also at breaking points: the new method is always introduced at a low activity level while the previously operated methods either keep the same activity levels (on intra-marginal lands which are not affected by an extension of cultivation) or reduce them slightly (in order to leave room to the introduction of a more intensive method on a fully cultivated land). The smooth adaptation of activity levels is a universal property which reduces the core of the dynamics to the identification of the outgoing marginal method and the incoming marginal method at critical moments: we call that phenomenon the law of succession of methods.

3 The law of succession of methods

3.1 The outgoing method

In case of extensive cultivation, the limit of an equilibrium is reached when some land becomes fully cultivated. The corresponding method of cultivation is the outgoing marginal method, i.e. it is marginal in the present equilibrium but will become intra-marginal in the next. Consider alternatively intensive cultivation: a land of a uniform quality is fully cultivated with, say, one barley method and two coexisting corn methods (the differences between Ricardo's and Sraffa's conceptions of the intensification process are inessential at this stage), one of them being more productive. When demand increases, the intensive corn method is progressively substituted for the other and the limit is reached when the less productive method is no longer operated. In all cases, the end of an equilibrium is defined by a physical constraint which allows us to identify the outgoing marginal method. All other presently operated industrial and agricultural methods (the barley method and the intensive corn method) will be operated in the next equilibrium.

3.2 Incoming method: physical vs. value criterion

The point on which Sraffa criticised Ricardo concerns the determination of the incoming method. Let us first follow Sraffa's critique concerning extensive cultivation. Ricardo assumed that lands can be classified according to their fertility, and the extension of cultivation follows that natural order. This is indeed the case if cultivation on a land of a lesser grade requires more of any input than on a better land, but the hypothesis is unduly restrictive. Sraffa substituted a value criterion for Ricardo's physical criterion and showed that the order of cultivation is dictated by costs of production, the cheapest lands being cultivated first. Since relative costs depend on distribution, the order may vary with it and is not given by Nature (*PCMC*, Section 86).

The economic literature has paid less attention to the fact that the same distinction between a physical and a value criterion underlies the difference between Ricardo's and Sraffa's conceptions of the intensification process. Ricardo (1817, Chapter 2) introduced that notion as follows:

"It often, and, indeed, commonly happens, that before No. 2, 3, 4, or 5, or the inferior lands are cultivated, capital can be employed more productively on those lands which are already in cultivation. It may perhaps be found, that by doubling the original capital employed on No. 1, though the produce will not be doubled, will not be increased by 100 quarters, it may be increased by eighty-five quarters, and that this quantity exceeds what could be obtained by employing the same capital, on land No. 3.[3] In such case, capital will be

[3] A few lines before that passage, Ricardo had assumed that the same amount of capital produces 100 quarters on land 1, 90 on land 2 and 80 on land 3. He assumes here that lands 1 and 2 are fully cultivated and compares the extension of cultivation on land 3 and its intensification on land 1.

preferably employed on the old land [...]".

In Ricardo's views, the intensification process consists in the investment of an additional layer of capital on an already fully cultivated land: more seeds, more manure and/or more labour are deposited on a part of that land. No supplementary rent is paid because that investment takes place on a fully cultivated land, for the use of which farmers have already paid: the overall rent remains the same when the part of land receiving manure is extended. By contrast, Sraffa characterises the intensification process by the coexistence of two agricultural methods, the intensive method being more productive per acre. To clarify the distinction between the two conceptions, let us formalise them for a corn model inspired by Ricardo's numerical example (even if, for simplicity, we ignore the difference between net and gross product). Let the initial method be

$$a_1 \text{ qr. corn} + l_1 \text{ labour} + 1 \text{ acre land} \rightarrow 100 \text{ qrs corn} \tag{1}$$

Sraffa considers that the intensification process consists in the coexistence of method 1 with another method 2 on the same land

$$a_2 \text{ qr. corn} + l_2 \text{ labour} + 1 \text{ acre land} \rightarrow 185 \text{ qrs corn} \tag{2}$$

while, for Ricardo, the additional investment (which takes place after method 1) per acre of land is written

$$\Delta a \text{ qr corn} + \Delta l \text{ labour} \rightarrow 85 \text{ qrs corn} \tag{3}$$

Both formalisations are equivalent on the peculiar hypothesis $a_2 \geq a_1$ and $l_2 \geq l_1$ ($\Delta a = a_2 - a_1 \geq 0$, $\Delta l = l_2 - l_1 \geq 0$). That hypothesis is analogous to the physical criterion sustaining the ranking of lands on a natural basis. No restriction of that type, however, is required when one refers to a value criterion, the only condition set by Sraffa to the coexistence of two methods being the nonnegativity of rent (*PCMC*, Section 87).

The distinction between the two criteria being clarified, it must be said that the attribution to Ricardo of a merely physical criterion proceeds from a simplification, as Ricardo made explicit references to values in many passages of the *Essay* and in the *Principles*. At a very general level, the identification of rents with differential costs means that only costs matter. More precisely, when Ricardo wrote: "The most fertile, and most favorably situated, land will be first cultivated, and the exchangeable value of its produce will be adjusted in the same manner as the exchangeable value of all other commodities, by the total quantity of labour necessary in various forms, from first to last, to produce it, and bring it to market" (*Principles*, Chapter 2), he clearly identified the quality of a land and the weakness of its overall costs of production, including the transportation costs. The *Essay* already made reference to equally fertile lands with different locations and showed that the overall quantity of capital per unit of product is the only magnitude which matters. Ricardo's most specific reference to costs as the ultimate criterion is found in the note attached to the last sentence of Chapter 2 (the same numerical example is referred to in Chapter 6). Here, Ricardo proceeds to an explicit determination of the order of cultivation and calculates the price of corn and the levels of rents. In that numerical example, the starting point is the productivity of labour in terms of additional corn:

"Let us suppose that the labour of ten men will, on land of a certain quality, obtain 180 quarters of wheat, and its value to be £4 per quarter, or £720; and that the labour of ten additional men will, on the same or any other land, produce only 170 quarters in addition; wheat would rise from £4 to £4 4 s. 8 d.".

"On the same land or any other" is a significant precision: the calculation concerns any type of rent. Since the productivity of labour decreases, the labour content of the last quarter increases by factor 180:170, and so does the price of corn from £4 to £4 4 s. 8 d. Then the rent on the previous land amounts to 180 - 170 = 10 quarters (the money rent rises at a higher rate than the corn rent because the price of corn itself rises). All calculations rely on costs only, and Ricardo's initial reference to a fertility criterion seems superfluous and even confusing. One may therefore reinterpret Sraffa's critique and consider that its relevance does not lie so much in the opposition between a physical and a value criterion as in Ricardo's reference to the labour theory of value: when prices are defined by labour contents, the relative costs are independent of distribution and therefore the order of cultivation seems to be given. (In the *Essay*, Ricardo had not yet developed the labour theory of value, but his reference to the notion of difficulty of production as the source of value led him to the same conclusion.) That illusion disappears when one refers to prices of production and, when transposed in that framework, the principles of Ricardo's calculations are safe.

3.3 The law of succession

When a scarcity constraint is met, prices and rents change. What is the law determining the incoming method? For the sake of simplicity, we retain Sraffa's hypothesis of a given rate of profit and assume that wages are paid *post factum*, though these assumptions are inessential.[4] Labour serves as the numéraire.

The general law is stated in three steps of increasing generality. Consider first a corn model with extensive cultivation. When the presently marginal land becomes fully cultivated, the price of corn starts rising. That rise improves the profitability of all methods on non-cultivated lands and stops when one of them yields the ruling rate of profit: the new marginal land and the new method are uniquely defined. Incidentally, the rents on fully cultivated lands also rise, but the phenomenon plays no role here.

Second, consider the choice between extension of cultivation on land 3 and its intensification on land 1, as in Ricardo's example, land 1 being already fully cultivated by one method. Everytime the price of corn increases by one shilling, the rent per acre on land 1 increases by α shillings, α being such that the rate of profit of the presently operated method on that land is maintained at its level. The rent on land 3 remains nil. The rise of corn improves the profitatibility of all methods on land 3 and, possibly, of some alternative methods on land 1 (those for which the positive effect due to the rise of corn supersedes the negative effect due to the rise in rent). The price of corn stops rising when some non-operated method yields the ruling rate of profit, and this determines the choice between extension and intensification.

Third, consider a basic bisector model with corn as the agricultural good

[4] Sraffa's hypotheses simplify the analysis because, at a given rate of profit, prices and rents are the solution to an affine system of equations. In particular, we shall use the following property. A price-and-rent equation being associated with each of the $\overline{n}$ operated methods, be they marginal or not, the present equilibrium price-and-rent vector x_0 is the solution of a linear system $Ax = b$ with $\overline{n}$ equations and $\overline{n}$ unknowns. Let us increase (more generally, change) demand and reach a breaking point. The outgoing marginal method once identified by a scarcity constraint, we delete the corresponding price equation, which leaves room for another equation (the one associated with the still unknown incoming method). Whatever the missing equation is, the new price-and-rent vector x_1 is of the type $x_1 = x_0 + \lambda\pi$, where x_0 is the previous price-and-rent vector while vector π, which represents the direction of the change in that vector, is entirely determined by the $\overline{n} - 1$ remaining methods. Therefore the only unknown magnitude is the scalar λ, the intensity of the change. Determining the incoming method and the new price-and-rent vector amounts to choosing the right value of λ.

and iron as the industrial good. Starting from a long-term equilibrium and its attached prices and rents, how is that equilibrium modified when a scarcity constraint is met? Ricardo stressed that the rise of corn implies that of rents ("Corn is not high because a rent is paid, but a rent is paid because corn is high", *Principles*, Chapter 2) but, in the passages of the *Principles* devoted to rent, he did not draw attention to the indirect effect on the prices of non-agricultural commodities. In the *Essay*, he even explicitly denied that effect, but he changed his opinion on that point soon after (see Sraffa's note in the *Essay* on that point and its references). The effect on other prices stems from interindustrial relationships and is explicitly mentioned in the chapters of the *Principles* relative to taxation: a rise of corn due to taxation (and the same if it stems from the introduction of a new marginal method) affects commodities into which corn enters directly and indirectly. A (lesser) compensating rise in those commodities is required to let them profitable. Assume a ruling rate of profit, as in Sraffa's formalisation. Any one-shilling rise in the price of corn entails a rise π in all prices and rents, where vector π is adjusted in order to maintain the profitability of the previously operated methods (the outgoing method apart) in their respective industries. These changes modify the profitability of all non-operated methods: some become less profitable, other more profitable. The general statement of the law is:

Law of succession of methods. Given an equilibrium and the evolution of demand, the outgoing method is determined by a scarcity constraint. The incoming marginal method is the first previously non-operated method which yields the ruling rate of profit when the price of corn rises, taking into account the effects on rents and all other prices.

The law defines the new long-run equilibrium in a unique way: were the rise in the price of corn (and in other prices and rents) smaller than the critical level defined by the law, there would be no incentive to introduce a new method; were it greater, the first method we are considering would yield more than the ruling rate of profit. The level of the rise is therefore the minimum compatible with the introduction of a new method. The law could alternatively be stated as a rule of minimum rise, which follows from competition between farmers. Ricardo's calculations in the already mentioned final note of Chapter 2 clearly illustrate the law of minimum rise.

The important lesson of the law is that the outgoing marginal method is determined by a physical side of the problem while the incoming method is determined by its value side. The next two Sections examine some consequences of that duality.

4 Reduction to single-product systems?

It was Ricardo's aim to extend the labour theory of value to production with lands, the labour values being defined by the operated methods in industry and the marginal methods in agriculture. The strategy is to get rid of rent by reducing the study of a productive system with land to that of a single-product system without lands.[5] This Section points at two independent limits of Ricardo's project and illustrates them by the violation of the trade-off property between wages and profits for a given level of demand. Note first that Ricardo's programme does work for extensive cultivation proper, when the following hypotheses hold: one agricultural good, one agricultural method on each quality of land, and given methods in industries (we shall return later on the last hypothesis, which has little to do with the intuitive content of notion of extensive cultivation and looks artificial). We first consider a corn model with intensive cultivation, then a multisector model. In both cases, the difficulties stem from the value criterion referred to in the law of succession of methods.

4.1 A corn model with intensive rent

In Section 3.2, we distinguished Ricardo's and Sraffa's conceptions of intensive rent. Method 1 being used on the totality of land, Ricardo imagines that a further layer of capital represented by method Δ is deposited on a part of that land. Since method Δ pays no rent, land can be ignored and the properties of a simple corn model without land hold. In Sraffa's more general case, the intensification of production is characterised by the coexistence of two methods (1) and (2). With labour as the numéraire, the price-and-rent equations associated with the simultaneous use of these methods are written

$$(1+r)a_1p + l_1 + \rho = 100p \qquad (4)$$
$$(1+r)a_2p + l_2 + \rho = 185p \qquad (5)$$

[5]It is assumed here that industrial methods are of the single-product type, otherwise the question of the 'reduction' is meaningless. Note however that the law of succession and all results of the paper apply to multiple-product systems.

A rent-free relationship is obtained by subtraction

$$(1+r)(\Delta a)p + \Delta l = 85p$$

With corn as the numéraire, the same equality is written as a relationship between the real wage w and the rate of profit r

$$(1+r)\Delta a + w\Delta l = 85 \tag{6}$$

If Δa or Δl are both positive (Ricardo's hypothesis), the rent-free equality (6) is the wage-profit relationship associated with the additional investment Δ described by relation (3) and the trade-off property is obvious. If Δa and Δl have opposite signs (Sraffa's generalisation), equality (6), which still holds, is not attached to a method of production, and it is immediately seen that the real wage and the rate of profit are positively correlated![6] Clearly enough, Sraffa did not see that consequence of his theory of intensive rent: Ricardo and Sraffa never cast doubt on the trade-off property between wages and profits, even if Ricardo stressed the community of interests of workers and capitalists against landlords ("It follows then, that the interest of the landlord is always opposed to the interest of every other class in the community", *Essay*; almost identical sentence in the *Principles*, Chapter 24).

4.2 A multisector model

As interindustrial relationships play an essential role in the other phenomenon we now study, we now consider a bisector model with corn and iron. When a scarcity constraint is met, the price of corn rises and also, as noticed above, that of iron. Let there be an alternative iron method which only uses small quantities of corn. Then the rise of corn has a negative but slight effect on its profitability, while the rise of iron has a positive impact. On the whole, the profitability of the alternative iron method improves, and that method participates in the run for profitability among non operated methods, as described by the law of succession. Suppose it wins the race and is the first to yield the normal rate of profit, ahead of any corn method. By the minimum rule, it is that iron method which will be operated in the next equilibrium. The new equilibrium is then characterised by the coexistence

[6] The example comes from Bidard (2014), in a discussion of Fratini's (2012) paper on intensive rent.

of two iron methods, with the progressive transfer of the production of iron from the previous method to the new one, and the substitution does increase the net product of corn since the new method is corn-saving. (The economic phenomenon at stake is not fanciful: a rise in the price of oil leads to the introduction of oil-saving methods in industry.) On the value side, however, the prices p_c of corn and p_i of iron are determined by both iron methods

$$(1+r)(a_{11}p_i + a_{12}p_c) + l_1 = p_i \tag{7}$$
$$(1+r)(a_{21}p_i + a_{22}p_c) + l_2 = p_i \tag{8}$$

The noticeable feature of that non-Ricardian equilibrium is that the conditions of production of corn do not intervene in the determination of prices, though corn is a basic commodity. Such prices have therefore no relationships with either labour values or prices of production *à la* Sraffa. As for corn, it is cultivated by means of a unique method on a fully cultivated land, and the rent is equal to the difference between the value of the crop and its overall cost of production, both values being calculated by means of the prices derived from system (7)-(8). Even if the usual two-step procedure (first the prices, next the rents) still applies, the Ricardian reduction to a single-product system fails. In particular, there are no analytical grounds for a trade-off property between wages and profits. (It is to discard non-Ricardian equilibria that the theory of extensive rent presumes that the industrial methods are given.)[7]

5 Dynamics and capital theory

5.1 Sraffa's argument

When the cultivation of corn is extended to a new land, the scarcity constraint at the origin of the change of methods is solved. Is that result also guaranteed if land is homogeneous, i.e. does the incoming method designated by the law of succession always help to increase the net product? The point deserves attention because the incoming method is chosen by means of a value instead of a physical criterion. We stress that the question is intrinsically linked to

[7] Bidard (2010) applied the dynamic approach to the theory of intensive rent proper, but determined the incoming method by considering the upper envelope of a family of curves. That procedure ignores the possible occurrence of non-Ricardian equilibria. This, incidentally, shows that the minimum rule may not result in a maximum wage property.

the dynamic approach and that Sraffa did examine it. Sraffa's argument (*PCMC*, Section 87) is that the positivity of rent ensures that the more expensive method is also more productive:

"If land is all of the same quality and is in short supply, this by itself makes it possible for two different processes or methods of cultivation to be used consistently side by side on similar lands determining a uniform rate per acre. While any two methods would in these circumstances be formally consistent, they must satisfy the economic condition of not giving rise to a negative rent: which implies that the method that produces more corn per acre should show a higher cost per unit of product, the cost being calculated at the ruling levels of the rate of profits, wages and prices."

To discuss the argument, consider a corn model with intensive rent and three methods of cultivation on a homogeneous land of total area 100 acres. Method 1 is land-intensive but the cheapest when rent is zero and its net product amounts to 40 quarters per acre. The productivity of method 2 is lower (20 quarters per acre), that of method 3 higher (60 quarters), but method 3 is costly because it is highly labour intensive. When final demand increases and reaches 4,000 quarters, the question of the incoming method arises. The obvious solution on the physical side consists in introducing the more productive method 3, not method 2. The economic problem stems from the application of the law of succession as it results from profitability considerations. Sraffa suggests, or seems to suggest, that the value criterion would indeed exclude the introduction of the less productive method 2 jointly with method 1, because it would lead to a negative rent. Though the 'economic condition' he states does hold at any given equilibrium, the argument does not apply to the succession of equilibria: the relative cost of methods 1 and 2 depends on the price-and-rent vector so that, even if method 1 is cheaper when rent is zero (that is why it is first used), it may well be more expensive than method 2 when both methods are operated jointly. This is what happens in the above example: when the price of corn rises, the law of succession leads to the introduction of method 2 rather than the more costly method 3. This does not result in a contradiction on the value side, as the intensive rent stemming from the coexistence of methods 1 and 2 is positive. The difficulty lies on the physical side: the coexistence of methods 1 and 2 does not solve the scarcity problem at the origin of the change of equilibrium, because method 2 which should be progressively substituted for 1 has a lower productivity per acre.

The phenomenon at stake is strongly connected with capital theory: for multisector systems without lands, there is no systematic relationship between profitability and productivity and, for instance, a fall in the rate of profit which, in a neoclassical approach, is deemed to favour the introduction of more capitalistic techniques, may not lead to an increase in the product per head. A similar conclusion holds in a corn-land model: the absence of an *a priori* connection between profitability and productivity explains why the incoming method defined by the law of succession may not solve the scarcity problem.

We arrive at a contradiction between the value side and the physical side of the problem. A first attempt to overcome it would be to modify the law of succession and to apply a minimum rule restricted to those methods which meet the physical constraint. The modified rule would imply that, once land is fully cultivated by method 1, method 3 is introduced. But the use of the labour-intensive method 3 requires a high price of corn, so high that the non-operated method 2 would yield extra-profits, so that the contradiction remains. A more systematic attempt is to proceed by drawing up the exhaustive list of all technical combinations (this is the way privileged in the static approach). As one can discard the joint use of three methods (the price of corn would be overdetermined) and that of methods 1 and 3 (method 2 would yield extra-profits), three possibilities remain:

(i) If only one method is operated, it must be the cheapest method 1. The long-term equilibrium E_1 sustained by that method can produce between 0 and 4,000 quarters.

(ii) The joint use of methods 1 and 2 sustains a long-term equilibrium, denoted E_{12} (method 3 is too costly at the associated price and rent). That equilibrium can produce between 4,000 and 2,000 quarters according to the proportions of the two methods.

(iii) The joint use of methods 2 and 3 sustains a long-term equilibrium E_{23} (the land-intensive method 1 is not profitable when rent is high) which can produce between 2,000 and 6,000 quarters.

It might seem that the last combination provides the solution to the scarcity problem when demand exceeds 4,000 quarters. This, we claim, only holds from the formal point of view retained in the static approach. The Ricardian dynamics follow the sequence of equilibria when the demand for corn increases from low to high levels. When it is low, method 1 is operated and progressively extended to the whole land, until demand reaches 4,000 quarters. A switch from equilibrium E_1 to E_{23} at that point would mean that the cultivation of the whole land by method 1 would suddenly be replaced by the joint use of methods 2 and 3, each on half of the land. That phenomenon is not consistent with Ricardo's views, which assumed a smooth physical transition at breaking points and the progressive introduction of one new method. The overall conclusion is that, in that example, the Ricardian dynamics fail.

5.2 An algebraic criterion

The condition for the working of the dynamics is that productivity goes with profitability. Let us give it an algebraic expression in the corn model defined by methods (1) and (2). Assume that method 1 is cheaper when demand is so low that rent is zero. Then the price p_1 of corn is solution to

$$(1+r)a_1p_1 + l_1 = 100p_1 \tag{9}$$

(with $100-(1+r)a_1 > 0$). Once land is fully cultivated by that method, 'the economic condition of not giving rise to a negative rent' (let us call it Sraffa's condition) when methods 1 and 2 are operated jointly is the existence of a nonnegative solution (p, ρ) to the system

$$(1+r)a_1p + l_1 + \rho = 100p \tag{10}$$

$$(1+r)a_2p + l_2 + \rho = 185p \tag{11}$$

Solving these equations in ρ gives the algebraic condition for the nonnegativity of rent (condition (12) below). Let us rather use an economic reasoning based on the dynamic approach. When land becomes fully cultivated, the price of corn rises and the rent becomes positive, these changes being such that method 1 maintains its rate of profit. The relationship between these rises is obtained by subtracting (9) from (10), from which we get $(p, \rho) = (p_1, 0) + \lambda\pi$, where $\pi = (1, \alpha) = (1, 100 - (1+r)a_1)$ is the direction of change and λ is a positive scalar: any one-shilling rise in corn implies an α-shilling rise in rent. Method 2 is introduced if its profitability, which is too low in the first equilibrium, improves when the price-and-rent vector varies in direction π (and the rent is then positive). Comparing the positive effect on the profitability of method 2 due to the price increase and the negative effect due to rent leads to condition

$$185 - (1+r)a_2 > 100 - (1+r)a_1 \tag{12}$$

That inequality is Sraffa's condition ensuring the nonnegativity of rent when both methods are operated jointly. It differs from the one ensuring that the incoming method has a higher productivity, which is written

$$185 - a_2 > 100 - a_1 \tag{13}$$

The lessons are:

- When the rate of profit is zero, the incoming method is always more productive and the Ricardian dynamics work.

- When it is positive, the working of the dynamics is submitted to the algebraic condition (E) that the two scalars $(100-(1+r)a_1)/(100-a_1)$ and $185-(1+r)a_2)/(185-a_2)$ relative to the consecutive techniques have the same sign.

The first statement expresses the duality property, or golden rule, between the quantity side and the value side when the rate of profit is zero. The second property is a further aspect of the so-called paradoxes in capital theory (see Harcourt, 1969, for an overview of the debates).

Figures 1 and 2 illustrate the discussion for three methods on homogenous land. Method 1 is operated when rent is zero (equilibrium E_1), then method 2 jointly with method 1 (equilibrium E_{12}) and finally method 3 jointly with method 2 (equilibrium E_{23}). On the horizontal axis the activity level of the incoming method is increasing. The vertical axis shows the corresponding net products. In Figure 1 the dynamics work: 'The increase [of production] takes place through the gradual extension of the method that produces more corn at a higher cost, at the expense of the method that produces less. As soon as the former method has extended to the whole area, the rent rises to the point where a third method which produces still more corn at a still higher cost can be introduced to take the place of the method that has just been superseded' (*PCMC*, Section 88). This is a precise description of the succession of methods *if* the incoming method is indeed more productive. Sraffa mistakenly thought that the nonnegativity of rent suffices to ensure that property, while the condition is of another nature. Figure 2 illustrates the example studied in Section 5.1 when the incoming method 2 is less productive than the method 1 it replaces progressively.[8] Then the Ricardian dynamic fail.

[8] In a corn-land model with homogeneous land, the productivity of a method is given by its net product per acre, and it is immediately seen which of two methods is more productive. For multisector models, the productivity of an agricultural method depends on the industrial methods with which it is associated: productivity is the inverse of the

(**See FIGURES 1 and 2 at the end of the paper**)

To put it differently, we agree with the idea that rent theory meets difficulties in the presence of several types of lands and several agricultural products. The reason is that the same problems are already there in case of a unique variety of lands and a unique agricultural product. No economic phenomenon seems to be specifically linked to the multiplicity of agricultural products.

6 The static approach

6.1 The static problem

Sraffa's work on the treatment of lands in long-term equilibria initiated many contemporary studies, starting from Quadrio-Curzio (1966). For extensive cultivation, Montani (1972, 1975) showed that the order of cultivation coincides with that of the (decreasing) wages sustained by the agricultural methods in the absence of rent. The treatment of intensive rent set more questions, some of them (e.g., is the intensive rent linked to a monopoly power of landowners?) being nowadays outdated. Vidonne (1977), Kurz (1978), Guichard (1979), Abraham-Frois and Berrebi (1980) and Klimovsky (1981) are representative of early researches. Though not stated explicitly, the idea that the complexities of rent theory are linked to intersectoral relationships prevented post-Sraffian authors to study the corn-land model (Freni (1991) is an exception) and favoured the use of heavy mathematical procedures, only partly alleviated by the use of geometrical figures. A more surprising point is that post-Sraffian scholars do not identify the dynamic approach as a significant component of Ricardo's and Sraffa's methodology and always adopt a static point of view: a long term-equilibrium is the solution of a system of equalities and inequalities for a given demand. Some of these equalities refer to the physical side (scarcity and demand constraints) and others to the value side (e.g., the uniformity of the rates of profit). The transformations of equilibria with demand play no role in that approach: this explains why the law of succession of methods, though at the core of the Classical

land content per quarter produced. One can define the 'land value' of a commodity as its direct and indirect content of land. Then an agricultural method is more productive if the land value of the corn it produces is smaller (Bidard, 2010). If lands are heterogeneous, one can extend the measure and define the lands values of a commodity.

approach, is not stated in that literature. As a matter of fact, the Ricardian dynamics are faithfully described in most books devoted to history of economic thought but, paradoxically, are missing in those attempting to develop modern versions of the Ricardian theory (e.g. Schefold, 1989; Kurz and Salvadori, 1995; Bidard, 2004) and from papers on rent theory (e.g., collective book edited by Bidard, 1987). (We have already noticed Bidard (2010) as a recent exception.) We see two main reasons for that puzzling situation: first, Sraffa's formalisation, undoubtedly a useful tool for a rigorous study of rent, remained incomplete because Sraffa did not write down the constraints relative to the 'requirements for use' (the same for multiple-product systems in general). The completion of the formalisation with the explicit introduction of a demand vector provided a valuable guide for further analysis. Second, that formalisation proved its efficiency by allowing the researchers to study new questions and also by casting doubts on some of Sraffa's statements: e.g., Saucier (1981) discovered the existence of non-Ricardian equilibria ('external differential rent', in his terminology) and D'Agata (1983) provided a first numerical example with multiple equilibria. The main critique we address to modern studies is to present themselves as faithful to Ricardo and Sraffa's approach, an appraisal which is at least partly disputable.

7 Existence and uniqueness

Existence and uniqueness are typical static problems, which were treated as such respectively by Salvadori (1986) and Erreygers (1990, 1995). We would like to reinterpret these results and to point at their connections with the Ricardian dynamics.[9] Technical details are given in Appendix B.

As soon as scarce resources are required for production, the levels of demand sustained by long-term equilibria admit an upper bound. Can an estimate of that limit be guessed directly from the initial data (list of methods, areas of lands and distribution)? This is an existence problem, as it amounts to ensuring the existence of an equilibrium when demand is smaller than a certain level. It was solved by Salvadori (1986) who transposed a mathematical result relative to linear complementarity problems, but that result

[9] Erreygers does not state the law of succession and does not consider demand as a parameter driving the change of equilibrium. His approach, however, has some common features with the Ricardian dynamics and, in particular, the study of neighbouring techniques differing by one method plays an essential role in his construction.

is disconnected from dynamics.[10] In the dynamic approach, the existence problem is linked to the law of succession of methods which, we recall, defines the incoming method as the first non-operated method which becomes profitable when the price-and-rent vector moves in a certain direction. Obviously, the rule only applies if the move improves the profitatibility of one method at least. It can be shown that this is indeed the case when demand is not too high, and the upper limit of the demand level thus found is the one which ensures the existence of a long-term equilibrium. (The bounded set D of demand vectors for which existence is ensured is formally defined by formula (17) in Appendix B.)

Global uniqueness means that any admissible level of demand is sustained by a unique long-term equilibrium. In the presence of several commodities, infinitely many trajectories lead from a low level of demand to a given demand vector, and the uniqueness property amounts to stating that the final equilibrium is path-independent. The connection between uniqueness and dynamics is clear in Figures 1 and 2 associated with a corn-land model. When the Ricardian dynamics work (Figure 1), the incoming method sustains an increase in production, and therefore there is a one-to-one correspondence between an equilibrium and a range of levels of demand. Figure 2 illustrates the opposite case: the dynamics fail because the incoming method in equilibrium E_{12} leads to a decrease of production and, then, intermediate levels of demand are sustained by multiple equilibria (three equilibria for $d = 3,000$). The analysis is more complex in multisector models. The general condition for local uniqueness (i.e., when comparisons are restricted to techniques which differ by one method α or β) is expressed as a sign equality between two magnitudes $e_\alpha = \delta_\alpha(r)/\delta_\alpha(0)$ and $e_\beta = \delta_\beta(r)/\delta_\beta(0)$. That condition, first stated by Erreygers (1990), is more easily found in a dynamic approach and its interpretation is clearer: as in equalities (12) and (13) above, the scalars involving r are relative to profitability, while the same expressions with $r = 0$ are relative to productivity. The assumed sign equality amounts to setting that both phenomena go together. Moreover, the criterion admits a global version (Erreygers, 1990, 1995): global uniqueness holds if and only if the Ricardian dynamics work everywhere, i.e. if the productivity condition

[10] Salvadori stressed that the mathematical proof is constructive, i.e. equilibrium is reached as the final step of an algorithm. That algorithm, however, admits no clear economic interpretation (or, at least, no attempt is made to give it an interpretation). In the next Section 8, we point at the connections between Ricardo's approach and another algorithm of the same family.

which generalises the above-mentioned condition (E) holds at every change of method. (See Appendix B for precise statements and a proof inspired by the dynamic approach.) That approach is also richer: it shows that the underlying cause of the multiplicity of equilibria at a given level d of demand is the failure of the Ricardian dynamics at some point when one tries to link a low-level equilibrium with an equilibrium of level d. (Sceptical readers are invited to check that statement on any numerical example of multiplicity given in the economic literature.)

8 Ricardo and modern mathematics

Ricardo is an unknown pioneer of a fruitful modern mathematical method commonly used to solve linear complementarity problems.

Complementarity problems are a family of problems frequently met in theoretical and applied mathematics: thousands of papers have been written on the topic in the last fifty years, and potentially every field of science has its own complementarity problems (Facchinei and Pang, 2003). The unknowns of a complementarity problem are nonnegative variables: in economics, they are typically activity levels and quantities on the physical side, prices on the value side. Complementarity means that the problem is expressed in terms of equalities (e.g., all operated methods yield the same rate of profit) and inequalities (e.g., the cultivated areas do not exceed the available areas) and, when some inequality is strict, the dual variable attached to it is zero (rent is zero on non fully cultivated lands). General equilibrium is the most famous complementarity problem in economics: the inequalities express that the excess supply on each market is nonnegative and, if the inequality is strict for some good, its price is zero. Though the existence of a solution can be dissociated from its calculation (e.g., the existence of a general equilibrium is proved independently of the convergence of the *tâtonnement* process, which may fail), the question of the effective determination of a solution remained open for a long time, even in the simplest case of linear complementary problems, i.e. when the equalities and inequalities are linear functions of the unknowns (Cottle *et al.*, 1992). A long-term equilibrium with lands is the solution of a bimatrix game, which is a specific type of linear complementarity problem (Salvadori, 1986).

It is in 1965 that Lemke found an algorithm to calculate a solution of a bimatrix game. Lemke's method is close to the famous simplex algorithm used in linear programming and the discovery drew immediately the specialists' attention. Several extensions and variants were soon found. One of its variants, called the parametric Lemke algorithm, consists in making the problem one considers depend on a parameter. Though the original Lemke algorithm and its parametric version were elaborated for mathematical purposes only, it turns out that the parametric method admits an economic interpretation when applied to the land problem. The parameter then considered is the demand vector d. The mathematical strategy to find an explicit solution for a given vector d consists in starting from a simple solution corresponding to another vector d_0 (for instance, a level of demand so low that the scarcity of lands can be ignored) and to follow the transformations of the initial solution when the parameter moves along a path joining d_0 to d. If the transfer works, the problem is solved for vector d. It is immediately recognised that the stategy used in the parametric Lemke algorithm coincides with Ricardo's dynamic approach. The reason of the mathematical efficiency of the method is that, most of the time, small changes in the demand vector only need minor adaptations of the solution (adaptation of activity levels). It is only at a finite number of points that, in mathematical terms, a 'change of basis' is required, the new basis being obtained by 'pivoting', a procedure involving a change in one 'basic variable', the new basic variable being identified mechanically by applying a minimum rule. In the economic interpretation, one operated method is changed when a scarcity constraint is met, and the change obeys the law of succession.

There exists, however, a difference between the Ricardian dynamics and the parametric algorithm, as mathematicians have recognised the existence of a potential difficulty in the working of the algorithm: the new basis may not allow to go further on the oriented path initially drawn from d_0 to d. In the land model, this occurs when the new method is less productive than the one it replaces: the dynamics stop at this stage. By contrast, the rule adopted by mathematicians lets the algorithm make a U-turn on the path ('antitone move'). In Figure 2, this means that the parametric Lemke algorithm starts by following path OA, reaches a local maximum (4,000 quarters) at point A, then continues by reducing demand along path AB (equilibrium E_2) down to 2,000 quarters, and eventually follows path BC (equilibrium E_3). The reduction of demand at an intermediate stage allows the algorithm to find solutions for high levels in a further step. The method is mathematically powerful, but the temporary reduction of demand has no economic interpretation.

9 Conclusion

Since its elaboration two centuries ago, the Ricardian theory of rent has been the subject of controversial readings. The present overview of the topic is in line with Sraffa's interpretation of Ricardo but shows that the economic phenomena are more complex than these authors themselves thought. Its main messages are: (i) the reaction of the economic system to a physical scarcity is dictated by the law of succession of methods, which only takes into account the evolution of prices and rents; (ii) the associated physical phenomena may be complex: a shortage of corn may lead to the introduction of a new iron method or to the cultivation of barley on a land already cultivated with oat; and all these reactions may, or may not, fit the evolution of final demand; (iii) the difficulties of rent theory occur even in simple frameworks; (iv) Sraffa did not draw all consequences of his analysis and did not see that the use of a value criterion opens the door to all complications of capital theory; finally, (v) the dynamic approach followed by Ricardo and Sraffa is richer than the static framework mainly adopted in the last fifty years.

10 Appendix A: Fallowing and dynamics

The physical limit which marks the end of an equilibrium when demand increases seems to take different forms: for extensive cultivation, it is reached when some land becomes fully cultivated (upper bound); for intensive cultivation with several agricultural products, when the activity level of some method vanishes (lower bound). That duality is rather awkward and we propose to refer to a unique criterion, which is a nonnegativity condition (lower bound). This is achieved by considering fallowing as an agricultural method.

Fallowing is a method with the following characteristics: with some area of land as the only input, its product consists in the same amount of land and no final good. The price equation associated with it shows that fallowing is operated only if rent is zero. With the convention regarding fallowing as an agricultural method, a non fully cultivated land is now seen as a land on which fallowing is operated, and a land which becomes fully cultivated as a land on which the activity level of fallowing vanishes. Then the universal criterion to identify the physical limit of an equilibrium is that the activity level of some operated method, be it fallowing or another method, drops to zero.

Considering fallowing as a specific method is consistent with the representation of the Ricardian dynamics as a process of substitution of a new method for another when demand changes. According to the usual approach, the extension of cultivation means that more and more lands are cultivated, with more and more operated methods. But when non-cultivated lands are considered as those on which fallowing is operated, the same phenomenon is interpreted as the replacement of fallowing by another method. The number of operated methods remains constant all along the dynamical process, equal to the sum of the total number of commodities and of lands ('squareness'). This is why the knowledge of the operated methods and distribution allows us to determine all prices and rents (including the zero rents). The only exception to squareness occurs at breaking points, when the activity level of some method vanishes and therefore the number of operated method falls by one of its usual value.

11 Appendix B: Formal results

Let there be m methods of production with constant returns, n produced commodities, k types of lands and homogenous labour. Method i ($i = 1,...,m$) makes use of a vector $a_i \in R_+^n$ of material inputs, a vector $\Lambda_i \in R_+^k$ of lands ($\Lambda_i = 0$ for industrial methods and Λ_i has but one positive component for agricultural methods, but these facts play no role) and an amount $l_i \in R_+$ of labour (it is assumed that labour is directly or indirectly necessary for the production of a net product). The product, obtained after one period, is a basket of commodities represented by the vector $b_i \in R_+^n$. These technical data can be stacked as (A, Λ, l, B), where A and B are matrices of dimension (n, m), Λ a matrix of dimension (k, m) and l is an m-vector. Let $\overline{\Lambda} \in R_+^k$ be the k-vector representing the available areas of the various qualities of lands. Under Sraffa's hypotheses, the rate of profit per period r is given ($r \geq 0$) and wages are paid at the end of the period.

Let the final demand be represented by a vector $d \in R^n$. The unknowns which characterise a long-term equilibrium sustaining that final demand are: (i) the vector $y \in R_+^m$ of activity levels of the various methods; (ii) the price vector $p \in R_+^n$ of commodities, with labour as the numéraire; (iii) the rent vector $\rho \in R_+^k$ per acre of the various types of lands. A long-term equilibrium is a solution to the system of linear inequalities with complementarity relationships

$$
\begin{aligned}
(B-A)y &\geq d \qquad [p] && (14)\\
\Lambda y &\leq \overline{\Lambda} \qquad [\rho] && (15)\\
(1+r)A^T p + \Lambda^T \rho + l &\geq B^T p \quad [y] && (16)
\end{aligned}
$$

Inequality (14) means that final demand is met by the net product of the economy, the goods in excess being zero-priced. Inequality (15) expresses the scarcity constraints on lands, with a zero rent on non-fully cultivated lands. Inequality (16) means that no method yields more than the ruling rate of profit, and that only methods which yield that rate may be operated. Formalisation (14)-(15)-(16) is the one referred to in Section 6.1. An alternative but equivalent formalisation consists in inserting free disposal and fallowing in the list of available methods: then inequalities (14) and (15) are transformed into equalities. The choice of the more adequate formalisation is merely a matter of convenience, but the analytical aspects of the dynamic approach are more easily managed in terms of equalities: the only significant difference between the static and the dynamic points of view lies in the way to solve the system, not in the way to write it down.

In the dynamic approach, an equilibrium corresponding to final demand d admits generically $n+k$ operated methods, the corresponding columns $(\widehat{A}, \widehat{\Lambda}, \widehat{l}, \widehat{B})$ describing the active part of the economy. When, as a consequence of a continuous change in vector d , the positive activity level of some method vanishes, one operated method must change according to the law of succession. In the light of that approach, let us examine the questions relative to existence, local and global uniqueness, and one more property (the proofs follow Bidard (2011)).

(i) The law of succession

Starting from a given long-run equilibrium with (p, ρ) as its (nonnegative) price-and-rent vector, we consider a continuous change $d(t)$ in the demand vector (after Ricardo, it is usual to refer to 'increases' in demand, but only 'changes' matter in the present construction) and the basic question concerns the existence of an incoming method when a critical demand d is reached because some activity level vanishes. Let the vector $\pi = (p', \rho')$ represent the direction of the change in the price-and-rent vector at that point: the change π is calculated in order that the profitability of all operated methods except the outgoing method is maintained (cf. Note 4, with $\overline{n} = n+k$) and, moreover, that the profitability of the outgoing method decreases (otherwise, replace π by $-\pi$). If π has some negative component, either a free disposal method or a fallowing method becomes profitable at $(p, \rho) + \lambda\pi$ for $\lambda > 0$ great enough. Assume that π is semipositive. Let $d(t)$ vary in the bounded set D defined as

$$D = \left\{d; \exists z > 0 \;\; d << (B - (1+r)A)z \text{ and } \Lambda z \leq \overline{\Lambda}\right\} \tag{17}$$

($<<$ means that the vector inequality is strict componentwise, $\leq$ that it is large, and $<$ that it is strict for one component at least) and assume for a moment that inequality

$$(B^T - (1+r)A^T)p' \leq \Lambda^T \rho' \tag{18}$$

holds. It follows by combining (17) and (18) that

$$d^T p' < z^T(B^T - (1+r)A^T)p' \leq z^T \Lambda^T \rho' \leq \overline{\Lambda}^T \rho' \tag{19}$$

Let y_0 denote the activity levels sustaining the equilibrium at a critical point d. Since inequalities (14) and (15) hold as equalities for $y = y_0$, the inequality between the extreme members of (19) is written

$$y_0^T(\widehat{B} - \widehat{A})p' < y_0^T \widehat{\Lambda}^T \rho' \tag{20}$$

Besides, equality

$$y_0^T(\widehat{B} - (1+r)\widehat{A})p' = y_0^T \widehat{\Lambda}^T \rho' \tag{21}$$

holds because the direction (p', ρ') of the change in the price-and-rent vector is

calculated in order to maintain the profitability of all methods with a positive activity levels at d. We conclude from the contradiction between (20) and (21) that inequality (18) does not hold, which means that the change in the price and-rent vector improves the profitability of some method. All methods having that property are non operated in the present equilibrium. The law of succession applies and the next equilibrium is well defined.

(ii) From the law of succession to existence

Let us draw an oriented path (C) from d_0 to d_1 in D. If the dynamics always worked, as Ricardo and Sraffa thought, there would be no difficulty to transfer an equilibrium at d_0 along that path and, by following its successive transformations, to define an equilibrium at d_1. The argument is more complex when U-turns and antitone moves are not excluded. It is based on the idea that, since finitely many equilibria can sustain a given demand (because finitely many combinations of methods are possible), the algorithm starting from d_0 will ultimately reach d_1 provided that it does not cycle, i.e. that the same equilibrium is not found twice. By construction, the following properties hold: given (C), an equilibrium sustaining d has uniquely defined entry and exit points (those for which some activity level vanishes; the entry and the exit point are permuted if one comes back on the same equilibrium during an antitone move), a uniquely defined successor (new equilibrium after the exit point) and a uniquely defined predecessor (previous equilibrium before the entry point).

Assume that the equilibrium sustaining d_0 is unique and let us denote the successive equilibria by letters A (the one sustaining d_0; by exception, A has no predecessor), B, C, etc., and assume that, in the sequence ABC... some letter is found twice. Consider the first letter K with that property. K cannot be A, otherwise the sequence would be AB...BA, and A would not be the first repeated letter. More generally, the sequence cannot be either KL...LK or KL...KL (because either L or J would be repeated before K) nor KLK (the predecessor and the successor of an equilibrium differ). Therefore loops are excluded, and the sequence of equilibria stops when point d_1 is reached.

To sum up, proving existence of an equilibrium for any d in D is reduced to establishing existence and uniqueness for some nondegenerate d_0 in D. In single-product systems, the idea would be to choose d_0 so small that no land is fully cultivated (we avoid $d_0 = 0$ which is degenerate because the activity levels $y = 0$ sustaining it have not $n+k$ positive components). A more radical choice consists in choosing d_0 in the negative orthant ($d_0 << 0$), by taking advantage of the fact that nothing in the previous proofs forbids that choice.

Lemma. For $d_0 << 0$, the only equilibrium sustaining d_0 is made of the n free disposal methods and the k fallowing methods, with $p = \rho = 0$.

If labour is required by all methods other than those referred to in the Lemma, the proof is immediate by using the complementarity relationships in relations (14)-(15)-(16). It extends easily to the case where labour is indispensable to obtain a nonnegative net product:

$$\{y > 0, (B - A)y \geq 0\} \Rightarrow l^T y > 0 \tag{22}$$

To sum up, the transfer of equilibrium works in D and the existence of a long-term equilibrium sustaining a final demand basket in that domain is ensured.

(iii) Local working of the dynamics

One must clearly distinguish the fact that an equilibrium admits a successor (the parametric Lemke algorithm works) from the working of the Ricardian dynamics: the Ricardian dynamics work when the transfer from one equilibrium to the next solves the physical problem which caused the loss of the first equilibrium. This is the case if the value side and the physical side go together. Local dynamics are concerned with the study of neighbouring techniques, i.e. which differ by one method only. The following two properties concern the value side and the physical side respectively, and hold for general multiple-product systems (Bidard, 2004).

Let us first prove a general property of the value side for neighbouring equilibria. The property we use is that method β (the one operated in the second equilibrium but not in the first) pays extra-costs at prices associated with the first equilibrium, and vice-versa for the method α operated in the first equilibrium but not the second.

Definition. For a given equilibrium, let $E(r)$ be the square $(n+k)$-matrix whose columns are the vectors $c_i(r) = \begin{pmatrix} b_i - (1+r)a_i \\ -\Lambda_i \end{pmatrix}$ for the operated methods i. The colour, white or black, of the equilibrium, is defined by the relative sign of $e(r) = \det E(r)$ and $e(0) = \det E(0)$.

The price-and-rent vector associated with an equilibrium satisfies equalities $(p^T, \rho^T)c_i(r) = l_i$. Two successive equilibria have $n+k-1$ methods in common and differ by one method only, say method α or method β (with $E_\alpha(r)$ and $E_\beta(r)$ as corresponding matrices). Relationships

$$\begin{aligned} (p'^T, \rho'^T)c_i(r) &= 0 \quad (i \neq \alpha, \beta) \\ (p'^T, \rho'^T)c_\alpha(r) &= a < 0 \\ (p'^T, \rho'^T)c_\beta(r) &= b > 0 \end{aligned} \tag{23}$$

mean that, by construction, the change (p', ρ') leaves the profitability of the common methods unchanged, decreases that of the outgoing method α and increases that of the incoming method β. Equalities $(p'^T, \rho'^T)(ac_\beta(r) - bc_\alpha(r)) = 0$ and (23) imply that $(p'^T, \rho'^T)(aE_\beta(r) - bE_\alpha(r)) = 0$, therefore matrix $aE_\beta(r) - bE_\alpha(r)$ has a zero determinant. As $E_a(r)$ and $E_\beta(r)$ differ by one column only, their respective determinants $e_\alpha(r)$ and $e_\beta(r)$ satisfy $ae_\beta(r) - be_\alpha(r) = 0$ and therefore have opposite signs:

$$e_a(r)e_\beta(r) < 0 \tag{24}$$

The second property concerns the quantity side of neighbouring techniques in general: do they sustain the net production of the same or of different baskets? (In the present framework, we would like that they produce different baskets.) The incoming method fits the move $d = d(t)$ only if some condition is met. Before the critical point, the demand vector $d(t)$ is positively generated by the columns of matrix $E_\alpha(0)$. The component of $d(t)$ on $c_\alpha(0)$ vanishes at the critical point and becomes negative at $d_+ = d(t_0 + \varepsilon)$. Since that component is equal to the ratio of the two determinants $\delta_+ = \det(..., c_i(0), ..., d_+)$ (for $i \neq \alpha$, i.e. for i varying in the set of the methods common to consecutive equilibria) and $e_\alpha(0) = \det(..., c_i(0), ..., c_a(0))$, these determinants have opposite signs. The Ricardian dynamics work locally if the same vector d_+ is positively generated by the columns of $E_\beta(0)$, i.e. if the ratio between δ_+ and $e_\beta(0)$ is positive. On the whole, the necessary and sufficient condition for that is

$$e_a(0)e_\beta(0) < 0 \tag{25}$$

That condition may or may not be satisfied. Since inequality (24) always holds, condition (25) can be given another form: let $\varepsilon_\alpha = \pm 1$ be the relative sign of $e_a(r)$ and $e_a(0)$. An equilibrium is dubbed 'white' if $\varepsilon_\alpha = 1$, and 'black' if $\varepsilon_\alpha = -1$. The condition for the local working of the Ricardian dynamics is that consecutive equilibria have the same colour.

If the Ricardian dynamics work, two consecutive equilibria do not sustain the same demand basket, hence the local uniqueness of the equilibrium; if they fail, the basket $d(t_0 - \varepsilon)$ is sustained by both the previous and the new equilibrium (non-uniqueness).

(iv) From the Ricardian dynamics to global uniqueness

Given any final demand vector d in D, is the equilibrium sustaining that demand unique, flukes apart? A necessary condition is that local uniqueness holds for any pair of neighbouring equilibria. Suppose that this condition is met everywhere in D and let us draw a path joining a demand vector d_0 to another d_1. Along that path, any equilibrium sustaining demand d_0 can be transferred and transformed by means of successive changes of methods into an equilibrium sustaining d_1. This is plain sailing, with no U-turn on the path. During the transfer, two distinct equilibria never merge (otherwise, by moving in the opposite direction, an equilibrium would have two successors, a contradiction with the law of succession). Since each equilibrium at d_0 generates a specific equilibrium at d_1, the number of equilibria at d_1 is not smaller than at d_0. As d_0 and d_1 have been chosen arbitrarily, the number of equilibria is constant on D. To show global uniqueness, i.e. for any d, it is therefore sufficient to show uniqueness for some d. The conclusion follows from the choice $d_0 << 0$ and the above lemma. To sum up, for any demand vector in D, the equilibrium is unique (and white) if and only if consecutive techniques have always the same colour or, which is the same, if the Ricardian dynamics work everywhere. [11]

(v) Number and type of equilibria

Let us draw an oriented path (C) in D, with no loop, starting from $d_0 << 0$, going to a given demand basket d and ending at $d_1 << 0$. In any case, the equilibrium at d_0 is transferred first to d, then to d_1, and the sequence of equilibria is uniquely defined. If the starting point is d_1, the same sequence is followed in reverse order. When the Ricardian dynamics work, the sequence of equilibria moves briskly along the path (Figure 3). If not, U-turns occur (Figure 4), and if the sequence comes back to point d, other equilibria sustaining that basket are found: those equilibria are called accessible by means of the path (C). It is clear from Figure 4 that the sequence of accessible equilibria sustains d an odd number of times. As each U-turn corresponds to a change of colour, the number of white accessible equilibria exceeds by one that of black accessible equilibria (at least generically, the points corresponding to a change of technique being excluded).

(**See FIGURES 3 and 4 at the end of the paper**)

Conversely, let us start from an arbitrary equilibrium sustaining basket d and let us make a step towards d_1. If the sequence of equilibria thus generated reaches either d_0 or d_1, the inital equilibrium is one of those which are reached on the unique trajectory from d_0 to d_1, i.e. it is accessible. If not, the alternative is that the sequence of equilibria makes a loop and comes back to its starting point. On a loop there are as many white as black equilibria, and loops are disconnected from each other (Figure 4). The overall conclusion is that, generically, the number of equilibria sustaining a given demand vector d in D is odd, the number of white equilibria exceeding by one that of black equilibria (Bidard, 2011). This result generalises a property previously established for joint production without scarce resources (Bidard and Erreygers, 1998) as well as Erreygers's (1990) original uniqueness result in the presence

[11] Erreygers's (1990, 1995) analysis, which introduced the global uniqueness property, is based on geometrical and topological considerations involving the possible existence of 'holes' and other intricacies which must be eliminated by an axiom. The present proof takes the existence domain D into account and requires no further condition.

of lands, viz.: if all equilibria sustaining d are of the same colour, there is in fact a unique equilibrium, which is white.

Theorem *For d moving in D, and flukes apart, the law of succession defines a unique successor to the present equilibrium. The number of equilibria sustaining a given final demand basket is odd, with the number of white equilibria exceeding by one that of black equilibria. The Ricardian dynamics work locally when two consecutive equilibria have the same colour. If this is the case everywhere in D, the Ricardian dynamics work everywhere and global uniqueness is ensured.*

To illustrate these results, let us come back to the numerical example given in Section 5.2. The first equilibrium is made of the joint use of method 1 and the fallowing method, the second equilibrium of the jointly operated methods 1 and 2. Their associated matrices as defined in point (iii) above are respectively:

$$E_\alpha(r) = \begin{bmatrix} 100-(1+r)a_1 & 0 \\ -\Lambda_1 & -1 \end{bmatrix}, E_\beta(r) = \begin{bmatrix} 100-(1+r)a_1 & 185-(1+r)a_2 \\ -\Lambda_1 & -\Lambda_2 \end{bmatrix}$$

In the absence of details relative to the available area $\overline{\Lambda}$ of land, the restriction stemming from condition (17) is written $100-(1+r)a_1 > 0$. We also assume that condition (12) is met, so that method 2 is indeed substituted for fallowing in the second equilibrium.

Since $e_\alpha(r) = \det E_\alpha(r) = -100+(1+r)a_1$, we have $e_\alpha(r) < 0$ and $e_\alpha(0) \leq e_\alpha(r) < 0$, therefore the first equilibrium is white ($\varepsilon_\alpha = 1$). Condition (12) being equivalent to $e_\beta(r) > 0$, the general property (24) is checked in that example. Eventually, the Ricardian dynamics work if and only if the new equilibrium has the same colour as the first, i.e. if $e_\beta(0) > 0$. This condition is indeed the one expressed by inequality (13). A more intriguing point is: let the data relative to method 2 be such that the dynamics do not work ($e_\beta(0) < 0$). The second method is less productive than the first, as in Figure 2 restricted to segments OA and AB (with no third method and no increasing final segment AB). For low levels of demand, there is one equilibrium, while for high lev... Ooops!

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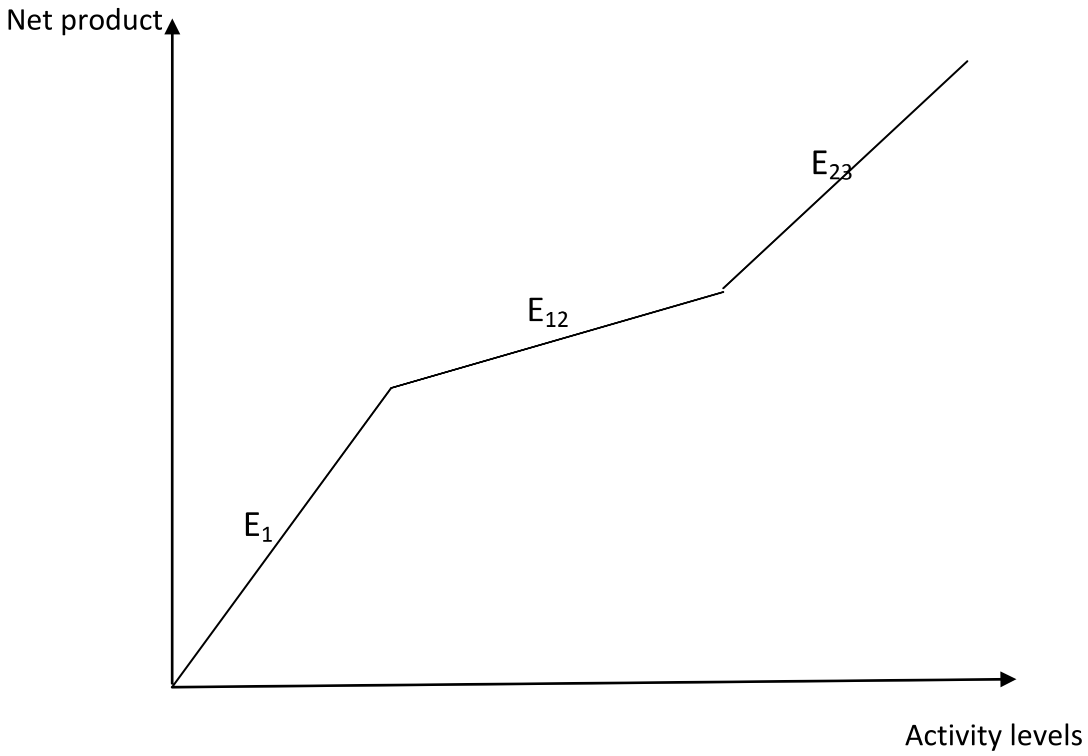

FIGURE 1. The Ricardian dynamics

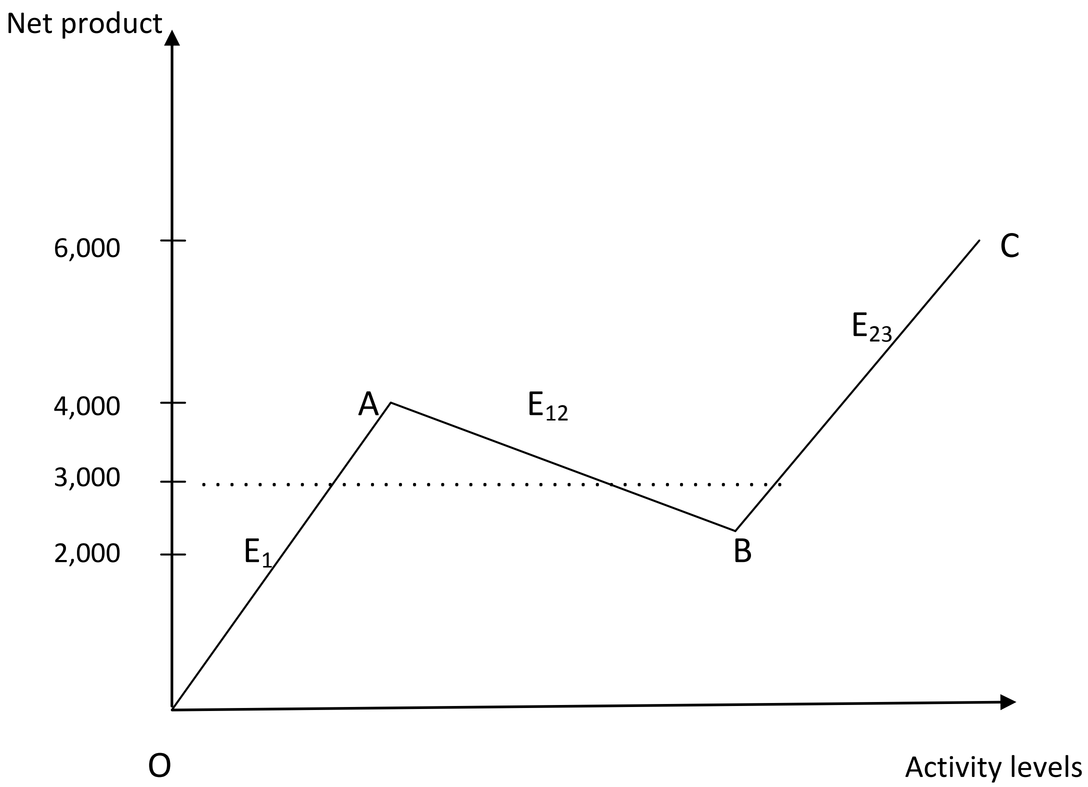

FIGURE 2. Failure of the dynamics

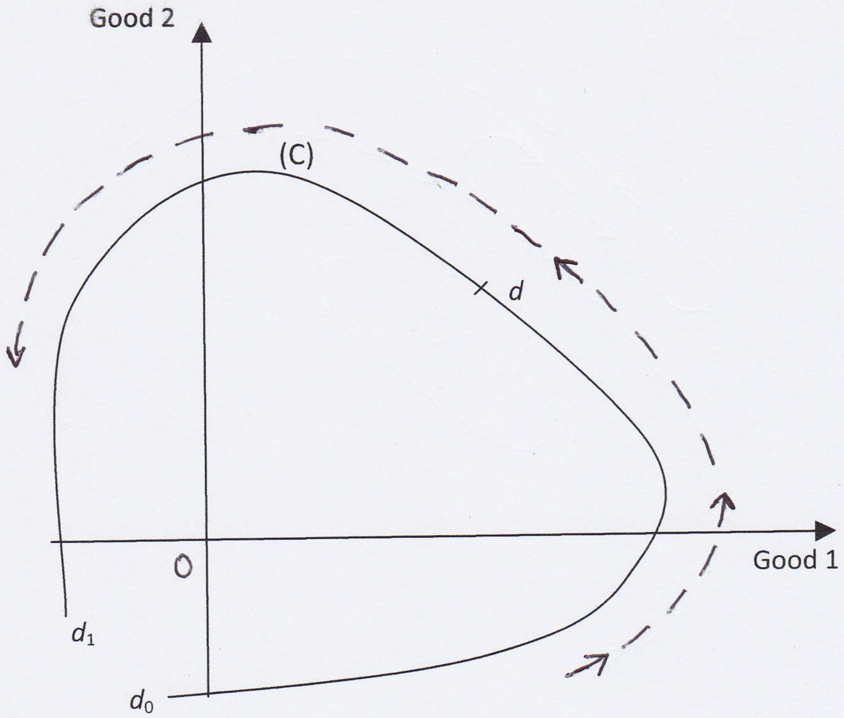

Figure 3. Uniqueness and Ricardian dynamics

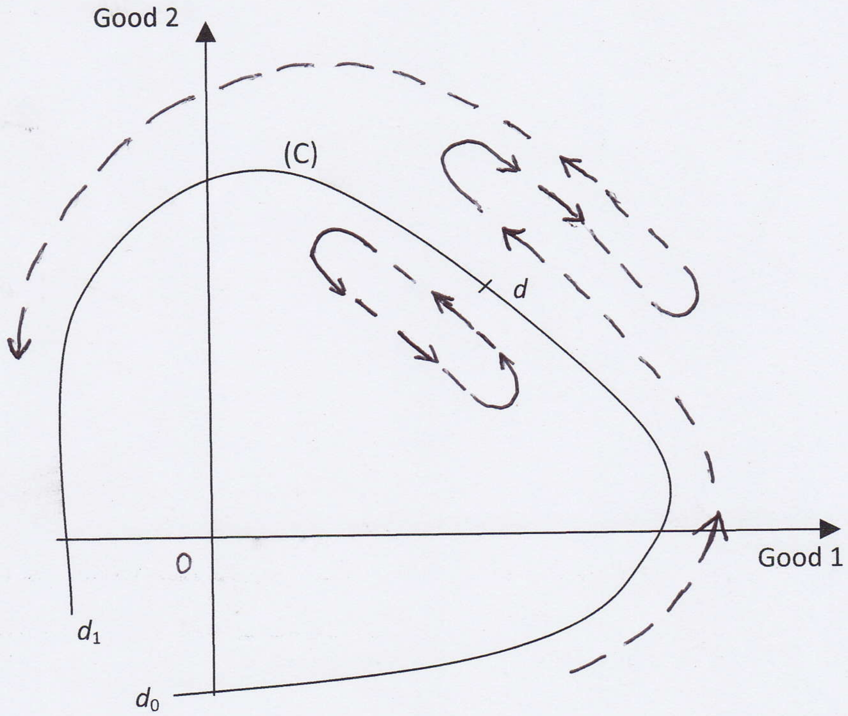

Figure 4. Five equilibria at *d*