The Euler Equation approach and utility functions: a critical view

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Abstract

The paper reviews the mathematical properties of the utility functions commonly used in the literature on the Euler equation, and discuss the implications of those properties in terms of individual preferences. It therefore considers the most relevant parameters characterizing utility functions, i.e. the coefficients of relative and absolute risk aversion, the intertemporal elasticity of substitution, the coefficients of relative and absolute prudence. It is shown that, despite measuring different aspects of preferences, the main parameters have close relationships between each other, which impose strict constraints on preferences. The analysis suggests that the choice of a specific functional form, which is crucial for the implications of the standard consumption model, is often made regardless of the features which make that function a plausible representation of individual preferences.

1. INTRODUCTION

The modern literature on consumption is based on the Life Cycle - Permanent Income Hypothesis developed by Modigliani and Brumberg (1954) and Friedman (1957). According to this model, consumers allocate resources over time in order to maximize lifetime utility subject to a budget constraint. Hall (1978) extended the model to the case of uncertainty with the introduction of the rational expectations assumption, and proposed to use the first order conditions of the intertemporal optimization problem faced by the consumer to derive a set of orthogonality conditions. The framework is that of an individual who maximizes the expected utility of consumption over a certain time horizon subject to an intertemporal budget constraint and a terminal condition on wealth. Assuming that consumers can borrow and lend at the same interest rate and that the utility function is state and time separable, one obtains the well-known Euler equation for consumption:

$$\boldsymbol{u'(C_t) = (1+\rho)^{-1}\mathbb{E}_t[(1+r_t)\cdot u'(C_{t+1})]}$$

where $\boldsymbol{\rho}$ is the rate of time preference and $\boldsymbol{r_t}$ is the interest rate.

The approach proposed by Hall, known as the Euler equation approach, allows to both test the validity of the model and to estimate some of the structural parameters of the utility function. Although it has been challenged several times, the Euler equation approach has become the standard approach to consumer behavior.

In its more general formulation, the standard framework encompasses many types of consumption behavior and has almost no testable implications. It is therefore necessary to construct a specific model which forces to make a number of strong assumptions and modelling choices. In particular, taking the model to the data requires to specify individual preferences.

Hall adopted a quadratic utility function and found that consumption is a martingale, but his result, highly influential on subsequent research, is due to the linearity of marginal utility associated with quadratic preferences. Hall's contribution proves that the choice of the utility function is crucial for the implications of the model in terms of consumer behavior.

The purpose of our analysis is to study the most relevant parameters characterizing utility functions, i.e. the coefficients of relative and absolute risk aversion, the intertemporal elasticity of substitution, the coefficients of relative and absolute prudence. In Section 2, after discussing the meaning of each structural preference parameter, we highlight the relations between them. We thus show that while measuring different aspects of preferences, the main parameters have close mathematical relationships between each other, which impose strict constraints on preferences. In Section 3 we provide a survey of the most popular utility functions adopted in the literature, emphasizing their main features and their shortcomings. The analysis suggests that the literature on the Euler equation and, specifically, the one devoted to estimating preference parameters, has been deeply affected by the constraints imposed on preferences by the utility functions adopted. Furthermore, the choice of a specific functional form often appears to be made regardless of the features which make that function a plausible representation of individual preferences.

2. PREFERENCE PARAMETERS

In this section we will describe the concepts of risk aversion, prudence and intertemporal elasticity of substitution and show that the standard model of consumer behavior involves close relations between them.

As for the empirical evidence on these parameters, several studies have attempted to estimate the magnitude of the coefficients of absolute and relative risk aversion, as well as the strength of precautionary motives and the degree of intertemporal substitution. Nevertheless, after decades of research, there appears to be little consensus regarding the magnitude of these parameters. As Carroll (2001) notices, «despite scores of careful empirical studies using household data, Euler equation estimation has not fulfilled its early promise to reliably uncover preference parameters»[1].

2.1 Risk aversion

An individual is risk averse if he or she is not willing to accept a fair gamble[2]. Hence, given the choice between earning the same amount of money through a gamble or through certainty the risk averse person will opt for certainty.

Analytically, if we assume two possible states of nature with consumption levels $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ and probabilities $\boldsymbol{\pi}$ and $(\mathbf{1}-\boldsymbol{\pi})$, then the utility of the expected value of consumption is:

$$\boldsymbol{u}(\mathbb{E}[\boldsymbol{c}]) = \boldsymbol{u}[\boldsymbol{\pi c_1} + (\mathbf{1}-\boldsymbol{\pi})\boldsymbol{c_2}]$$

and the expected utility of consumption is:

$$\mathbb{E}[\boldsymbol{u}(\boldsymbol{c})] = \boldsymbol{\pi u}(\boldsymbol{c_1}) + (\mathbf{1}-\boldsymbol{\pi})\boldsymbol{u}(\boldsymbol{c_2}).$$

An individual is risk averse if he or she would prefer the certain amount $\mathbb{E}[\boldsymbol{c}]$ rather than the expected utility, or, equivalently, if $\boldsymbol{u}(\mathbb{E}[\boldsymbol{c}]) > \mathbb{E}[\boldsymbol{u}(\boldsymbol{c})]$.

The risk attitude is directly related to the curvature of the utility function: if $\boldsymbol{u}(\mathbb{E}[\boldsymbol{c}]) > \mathbb{E}[\boldsymbol{u}(\boldsymbol{c})]$ then the utility function is concave and shows diminishing marginal utility[3].

[1] We will not describe the huge flow of research devoted to estimating preference parameters because it is beyond the purpose of the analysis: we wish to suggest that the literature on the Euler equation has been deeply affected by the constraints imposed on preferences by the utility functions adopted, rather than to evaluate the empirical results that literature achieved. See Browning and Lusardi (1996) for a discussion of the early literature and Attanasio and Weber (2010) and Jappelli and Pistaferri (2010) for more recent surveys. For a discussion of the econometric problems affecting Euler equation estimation see Ludvigson and Paxson (2001) and Carroll (2001).

[2] A fair gamble has an expected return of zero.

[3] If the utility function is concave the utility of the expected value of an uncertain amount is greater than the expected utility of that amount.

The modern literature on risk aversion began with the work of Pratt (1964) and Arrow (1965), who proposed formal measures of risk aversion[4].

The Arrow–Pratt measure of absolute risk aversion is:

$$\boldsymbol{A(c) = -\frac{u''(c)}{u'(c)}}$$

The Arrow–Pratt measure of relative risk aversion is:

$$\boldsymbol{a(c) = -\frac{u''(c) \cdot c}{u'(c)}}$$

The concept of absolute risk aversion is suited to the comparison of attitudes towards risky projects whose outcomes are absolute gains or losses from current wealth (or consumption), whereas the concept of relative risk aversion is useful to evaluate risky projects whose outcomes are percentage gains or losses of current wealth (or consumption).

It has been agreed widely in the literature that an individual's utility is plausible to exhibit decreasing (or at most constant) absolute risk aversion[5]: it is indeed a common contention that wealthier people are willing to bear more risk than poorer people.

The behavior of relative risk aversion is more problematic: the property of decreasing relative risk aversion implies that individuals become less risk averse with regard to gambles that are proportional to their wealth as their wealth increases. This is a stronger assumption than decreasing absolute risk aversion: a risk averse individual with decreasing relative risk aversion will exhibit decreasing absolute risk aversion, but the converse is not necessarily the case[6]. Assuming constant relative risk aversion is thought to be quite plausible an assumption.

2.2 Prudence

When preferences exhibit prudence, income uncertainty reduces current consumption, and thus raises saving: prudence leads individuals to treat future uncertain income cautiously and

[4] Since the degree of risk aversion depends on the curvature of the utility function, it is natural to measure it by means of its second derivative. But the risk attitudes are unchanged under affine transformations of $\boldsymbol{u}(\cdot)$, so the second derivative $\boldsymbol{u''}(\cdot)$ is not an adequate measure of risk aversion. Instead, it needs to be normalized. This leads to the definition of the Arrow–Pratt measures.

[5] The most straightforward implications of increasing or decreasing absolute or relative risk aversion occur in the context of forming a portfolio with one risky asset and one risk-free asset. If the individual experiences an increase in wealth, he or she will choose to increase the number of euro of the risky asset held in the portfolio if absolute risk aversion is decreasing. Similarly, if the individual experiences an increase in wealth, he or she will choose to increase the fraction of the portfolio held in the risky asset if relative risk aversion is decreasing.

[6] Since $\mathbf{a}(\boldsymbol{c}) = \boldsymbol{c} \cdot \boldsymbol{A}(\boldsymbol{c})$ we have that $\mathbf{a}'(\boldsymbol{c}) < 0$ implies $\mathbf{A}'(\boldsymbol{c}) < \mathbf{0}$.

not to spend as much currently as they would if future income were certain. The saving that results from the knowledge that the future is uncertain is known as precautionary saving.
Leland (1968) demonstrated that precautionary saving requires convex marginal utility[7] in addition to risk aversion, and Kimball (1990) proposed two measures of the intensity of the precautionary saving motive: a measure of absolute prudence:

$$\boldsymbol{P(c) = -\frac{u'''(c)}{u''(c)}}$$

and a measure of relative prudence:

$$\boldsymbol{P(c) = -\frac{u'''(c) \cdot c}{u''(c)}}$$

2.3 Intertemporal elasticity of substitution

According to the standard model, the consumption path over life cycle should be tailored to take advantage of the interest rate: if the interest rate between t and $t + 1$ is high, then there is an additional incentive to postpone consumption in t in favor of $t + 1$[8].
The magnitude of the response of consumption growth to changes in the interest rate is equal to the intertemporal elasticity of substitution and depends (inversely) on the curvature of the utility function:

$$\boldsymbol{\sigma = -\frac{u'(c)}{c \cdot u''(c)}}$$

The intertemporal elasticity of substitution thus represents the proportional change in consumption that must follow to a one percent change in the interest rate in order to keep the marginal utility of expenditure constant.

[7] The positive third derivative of the utility function, that is, the convexity of marginal utility, increases consumers' desire to insure themselves against the fall in consumption that would result from a fall in income and so increases the consumers' savings.
[8] The Euler equation in the case of a constant interest rate predicts that consumption raises over time if r exceeds ρ and falling if r is less than ρ:

$$\boldsymbol{u'(C_t) = \frac{1+r}{1+\rho}\mathbb{E}_t[u'(C_{t+1})].}$$

2.4 The relationship between preference parameters

The degree of risk aversion, the intensity of the precautionary saving motive and the degree of intertemporal substitutability, describe different aspects of individual preferences:

- o the measures of risk aversion tell us how much the individual is willing to sacrifice his consumption in the best case scenario in order to achieve a higher level of consumption in the worse;
- o the magnitude of prudence tells us how an increase in uncertainty about future income will affect current consumption;
- o the intertemporal elasticity of substitution tells us to what extent the individual is willing to substitute future consumption for current consumption in response to the incentive given by interest rates.

However, between the different preference parameters exist close relationships.

As Kimball and Weil (2009) show, in the standard model, which is to say under intertemporal expected utility maximization, there is an identity linking prudence to risk aversion:

$$\boldsymbol{P(c) = A(c) - \frac{A'(c)}{A(c)}}$$

In addition, for utility functions which exhibit prudence, the coefficient of absolute risk aversion must be non increasing:

$$\mathrm{P(w)} > 0 \rightarrow A'(\mathrm{c}) \leq 0$$

Furthermore, Drèze and Modigliani (1972) prove that decreasing absolute risk aversion leads to a precautionary saving motive stronger than risk aversion:

$$\mathrm{A'(c)} < 0 \rightarrow \mathrm{A(c)} < \mathrm{P(c)}.$$

The link between risk aversion and intertemporal substitution is even closer. The simultaneous additivity induced by intertemporal separability and expected utility (additivity over periods and over states of nature) implies, indeed, a negative relation between risk aversion and intertemporal substitution: individuals who are risk averse will be unresponsive to intertemporal incentives, while those who are willing to reallocate their consumption in response to changes in the interest rate will display relatively little aversion towards risk.

3. UTILITY FUNCTIONS

In this section we provide a survey of the most popular utility functions adopted in the literature on the Euler equation, i.e. quadratic, exponential and isoelastic preferences, describing their main features and their shortcomings.

In Table 3.1 we present, for each functional form, the marginal utility, the coefficients of absolute and relative risk aversion, the coefficients of absolute and relative prudence, and the elasticity of intertemporal substitution.

TABLE 3.1

UTILITY FUNCTIONS

	u'(c)	A(c)	a(c)	P(c)	p(c)	σ
QUADRATIC $u(\boldsymbol{c}) = a\boldsymbol{c} - \frac{b}{2}\boldsymbol{c}^2$	$a - b\boldsymbol{c}$	$\frac{b}{a - b\boldsymbol{c}}$ IARA	$\frac{b}{a - b\boldsymbol{c}}\boldsymbol{c}$ IRRA	0	0	$\frac{a - b\boldsymbol{c}}{b\boldsymbol{c}}$
EXPONENTIAL $u(\boldsymbol{c}) = -\frac{e^{-\alpha c}}{\alpha}$	$e^{-\alpha c}$	α CARA	$\alpha\boldsymbol{c}$ DRRA	α	$\alpha\boldsymbol{c}$	$\frac{1}{\alpha\boldsymbol{c}}$
ISOELASTIC $u(\boldsymbol{c}) = \frac{\boldsymbol{c}^{1-\gamma}}{1-\gamma}$	$\boldsymbol{c}^{-\gamma}$	$\frac{\gamma}{\boldsymbol{c}}$ DARA	γ CRRA	$\frac{1+\gamma}{\boldsymbol{c}}$	1 + γ	$\frac{1}{\gamma}$
LOGARITHMIC $u(\boldsymbol{c}) = log(\boldsymbol{c})$	$\frac{1}{\boldsymbol{c}}$	$\frac{1}{\boldsymbol{c}}$ DARA	1 CRRA	$\frac{2}{\boldsymbol{c}}$	2	1

3.1 Quadratic Preferences

Quadratic preferences imply linear marginal utility: expected marginal utility of consumption is the same as the marginal utility of expected consumption. This entails that the third derivative of the function is zero, that is to say, quadratic preferences exhibit no prudence, so uncertainty about future income has no impact on consumption.

Earlier attempts at testing the standard model of consumption relied on the special case of quadratic preferences. This case is known in the literature as the Permanent Income model with certainty equivalence (Hall, 1978; Flavin, 1981; Campbell, 1987).

Under this assumption the Euler equation

$$\mathbb{E}_{\mathbf{t-1}}[\mathbf{u}'(\mathbf{C_t})] = \mathbf{u}'(\mathbf{C_{t-1}})]$$ [9]

[9] Following most of the literature, we are assuming a constant interest rate equal to the rate of time preference.

rewrites as:

$$\mathbf{C_t = C_{t-1} + \varepsilon_t}$$

where $\boldsymbol{\varepsilon}_\mathbf{t} = \mathbf{C_t} - \mathbb{E}_{\mathbf{t-1}}[\mathbf{C_t}]$ is a consumption innovation, i.e., the effect on consumption of all new information about the sources of uncertainty faced by the consumer. Hence, in the special case of quadratic preferences, consumption behaves as a martingale: ex ante current consumption is the best predictor of next period's consumption; ex post, consumption changes only if expectations are not fulfilled.

Quadratic utility implies certainty equivalence: the consumption function is the same as under certainty once expectations are replaced by realizations. The individual consumes the amount he or she would if his or her future incomes were certain to equal their means. Therefore in the certainty equivalence case there is no precautionary saving, so the model can be very misleading in the presence of uncertainty.

Quadratic utility implies increasing absolute risk aversion, so the amount of consumption that individuals are willing to give up to avoid a given amount of uncertainty about the level of consumption rises as they become wealthier. This property is unappealing on theoretical grounds and strongly counterfactual (riskier portfolios are normally held by wealthier households).

Quadratic preferences also imply that the willingness to substitute over time is a decreasing function of consumption, which is another feature that makes its use unappealing: poor consumers should react much more to interest rate changes than rich consumers, which is not the case.

3.2 Exponential Utility

Exponential utility has been widely used in the literature on consumption, such as in the well-known model proposed by Caballero (1990) where a closed form solution for consumption is derived.

This functional form is unique in exhibiting constant absolute risk aversion (CARA). Relative risk aversion is, on the contrary, increasing in consumption, which is quite unrealistic an assumption. Furthermore, as in the quadratic preferences case, intertemporal elasticity of substitution is decreasing.

These features make exponential utility an implausible representation of individual preferences.

3.3 Isoelastic Preferences

Isoelastic (or CRRA, or power) utility[10], a very popular preference specification, has been used in the consumption literature since the papers by Hansen and Singleton (1982 and 1983) and it has played a preeminent role in many theoretical studies of asset pricing (Merton, 1973; Rubinstein, 1976). In fact, the assumptions of CRRA utility and lognormality of the joint distribution of consumption and stock returns together lead to an empirically tractable, closed-form characterization of the restrictions implied by the model.

However, such a specification also imposes strong restrictions on preferences: the elasticity of intertemporal substitution of consumption is, in this context, constant and equal to the reciprocal of the degree of risk aversion. Hence, the curvature parameter $\boldsymbol{\gamma}$ plays a double role. On the one hand it is equal to the coefficient of relative risk aversion and therefore summarizes consumer's attitude towards risk. On the other, its reciprocal is equal to the elasticity of intertemporal substitution and therefore measures as consumption growth changes when the relative price of present and future consumption changes.

In the CRRA case there is also a close link between intertemporal substitution and prudence. It follows because we have only one parameter in the utility function so this must control all the aspects of preferences.

4. CONCLUSIONS

In the preceding sections we have considered the mathematical properties of the utility functions commonly used in the consumption literature and discussed the most relevant parameters characterizing individual preferences.

We have shown that the choice of a specific functional form can be crucial for the results obtained, like in the case of Hall's certainty equivalent model, which had a significant influence on subsequent research.

It could be argued that the choice of a specific functional form is often made regardless of the features which make that function a plausible representation of individual preferences. For example, the wide use of quadratic and CARA preferences is perhaps due, at least partially, to the fact that these utility functions allow the derivation of a closed form solution for consumption. As we have shown, though, both these functional forms are unappealing on theoretical grounds. As for CRRA utility, which is definitely the most popular function in

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10 Logarithmic utility is a CRRA function in which $\boldsymbol{\gamma}$ is equal to unity: $\boldsymbol{lim_{\gamma\to 1}\frac{{C_t}^{1-\gamma}-1}{1-\gamma} = log(C_t)}$.

consumption literature, its main advantage is analytic convenience, as it yields first order conditions that are log-linear in consumption. However, such a specification imposes strong restrictions on preferences because it has only one parameter so this must control prudence, risk aversion and intertemporal substitution.

The analysis suggests that the literature on the Euler equation and, specifically, the one devoted to estimating preference parameters, has been deeply affected by the constraints imposed on preferences by the utility functions adopted.

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