Centro di Ricerche e Documentazione "Piero Sraffa"

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Centro Sraffa Working Papers
n. 54

April 2022

ISSN: 2284-2845
Centro Sraffa Working Papers
[online]

Some recent developments on the explanation of the empirical relationship between prices and distribution

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March 30, 2022

Abstract

The paper complements recent contributions towards the explanation of the regularities in the behaviour of prices and capital intensities as an effect of hypothetical changes in the rate of profit in empirical production-price models. It is shown that theoretical price and capital curves, i.e. prices and capital values as a function of the rate of profits, depend on the product of the eigenvalues and what we call the eigenlabours —the representation of the labour vector in the space spanned by the eigenvectors of the input matrix. We report robust evidence that the eigenvalues by themselves cannot produce the curves regularly reported in the literature, but rather it is the joint action of the eigenvalues and the eigenlabours. It is conjectured that the tendency towards zero of the subdominant eigenlabours is driven by the statistical tendency towards the proportionality between the labour vector and the Perron-Frobenius eigenvector of the input matrix.

Keywords: Sraffian price models; price-profit rate curves; capital value; spectral representation; labour vector-Perron-Frobenius eigenvector relation.
JEL Class: B51; C67; D57.

> "The key to the movement of relative prices consequent upon a change in the wage lies in the inequality of *the proportions in which labour and means of production* are employed in the various industries. It is clear that if the proportion were the same in all industries no price-changes could ensue, *however great was the diversity of the commodity-composition of the means of production in different industries*." (Sraffa, 1960, p. 12, our emphasis)

1 Introduction

A growing literature has emerged over the last decade seeking to *explain* a set of stylised facts obtained from the computation of production-price models using information from input-output accounts of actual economies. Across countries and years, industry price indices maintain a simple relationship with income distribution. Empirical price curves, i.e., price as a function of the rate of profit, are more often than not monotonic and are well described by linear or quadratic approximations to the full curves. Industries' capital intensities and capital-output ratios are in most cases nearly linear. Thus, 'capital', i.e., the value of produced commodities used as means of production, bears the same simple empirical relationship with distribution. In addition, production prices computed at the observed profit rate and direct prices (prices proportional to total quantities of labour) are remarkably close to each other across different scalar measures.[1]

Given the potentially complex relationship between value and the profit rate in the Sraffian price model and the lack of *a priori* knowledge to further constrain the techniques of production (e.g. Schefold, 1976; Pasinetti, 1977, ch. V; Bidard, 2004, ch. 6),[2] the stylised facts in empirical price models are unexpected and constitute a paradox within the literature. These empirical regularities have been connected to important debates in the theory of value, distribution, and capital such as (i) the empirical likelihood and relevance of the capital paradoxes discussed in the Cambridge Capital Controversies and (ii) the accuracy of Marx's algorithm to obtain production prices from labour values and the conditions under which profits correspond to surplus value in long-period positions.[3]

[1] These results are robust to the inclusion of fixed capital matrices and the aggregation detail of industrial classification systems. Another regularity in empirical production-price models is the near linearity of the wage curve. Because wage curves in terms of the net or gross output involve information not only from the techniques of production but also from the social output composition, the constraints advanced in this paper to explain the regularities in price and capital curves cannot *fully* explain the near linearity of wage curves. Mariolis and Tsoulfidis (2016b, chap. 3), Tsoulfidis (2021, chaps. 3 and 4), and Shaikh (2016, chap. 9) provide a detailed description of the literature.

[2] The typical constraints in the techniques of production (the nonnegativity, productivity, and indecomposability, and primitivity of the relevant input matrix) are not sufficient to constrain the curves.

[3] Han and Schefold (2006) and Kersting and Schefold (2021) argue that these regularities imply a low likelihood of capital paradoxes and a low degree of substitutability. Petri (2021) and Zambelli (2018) argue otherwise. Others argues that the near-linearity of the price curves represent

There has been important contributions to the *identification* of the characteristics of the productive structure behind these regularities in price and capital intensity curves. One group of writers[4] has focused on the characteristics of the input-coefficient matrices $\mathbf{J} \equiv \frac{1}{\lambda_{\mathbf{H}1}}\mathbf{H}$ and proposes the hypothesis, i.e., the constraint on the productive structure that explains the stylised facts, of sufficiently small subdominant eigenvalues, $\lambda_{\mathbf{J}2} \approx \lambda_{\mathbf{J}3} \approx \cdots \approx 0$, where $\lambda_{\mathbf{J}1} = 1$ is the Perron-Frobenius (P-F) eigenvalue.[5] This constraint implies a strong proportionality in the columns of matrix $\mathbf{J}$.[6] They show that approximately linear price and capital intensity curves are obtained under sufficiently small subdominant eigenvalues. To lend support to their hypothesis, they study the eigenvalues of matrices for different countries and years and find that in every case most of the eigenvalues tend to cluster around zero with a histogram and rank-plot of their moduli showing a fast rate of decay.[7]

Closer examination of the computed eigenvalues however raise questions about this hypothesis. Most of the eigenvalues do cluster around zero, as shown by the frequency distributions of the eigenvalues' moduli $|\lambda_{\mathbf{J}k}|$ for eight matrices $\mathbf{J}$ for the U.S. economy in plot (a) of Figure 1. Nevertheless, there is an important *number* of subdominant eigenvalues with *considerable* magnitude: The cumulative count in plot (a), Figure 1 shows that on average 15 (between 11-20) subdominant $|\lambda_{\mathbf{J}k}| \geq 0.25$ in each year. In Mariolis and Tsoulfidis (2018, p. 13), the magnitude of the second and third eigenvalue ($|\lambda_{\mathbf{J}2}|$, $|\lambda_{\mathbf{J}3}|$) of China, Germany, and the U.S. are (.41, .31), (.53, .40), and (.48, .48), respectively. Plot (b) in Figure 1 shows that for the 645 economies (43 countries in 15 years) of the World Input-Output Database (WIOD) the percentage of subdominant eigenvalues with $|\lambda_{\mathbf{J}k}|/\lambda_{\mathbf{J}1} > 0.25$ fluctuate between 10%-12%. On average, there are 52 industries per economy, so these percentages correspond to 5.2-6.2 observations per economy, on average.[8] These numerous subdominant $\lambda_{\mathbf{J}k}$ with considerable magnitude can 'activate' the nonlinear terms in the

the empirical strength of the labour theory of value (e.g., Shaikh, 1998) and the accuracy of Marx's algorithm to derive production prices (Shaikh, 2020, sec. 5). Schefold (2016) derives a new set of conditions under which the economy's surplus value coincide with profits, which overlaps with those required to produce linear wage curves.

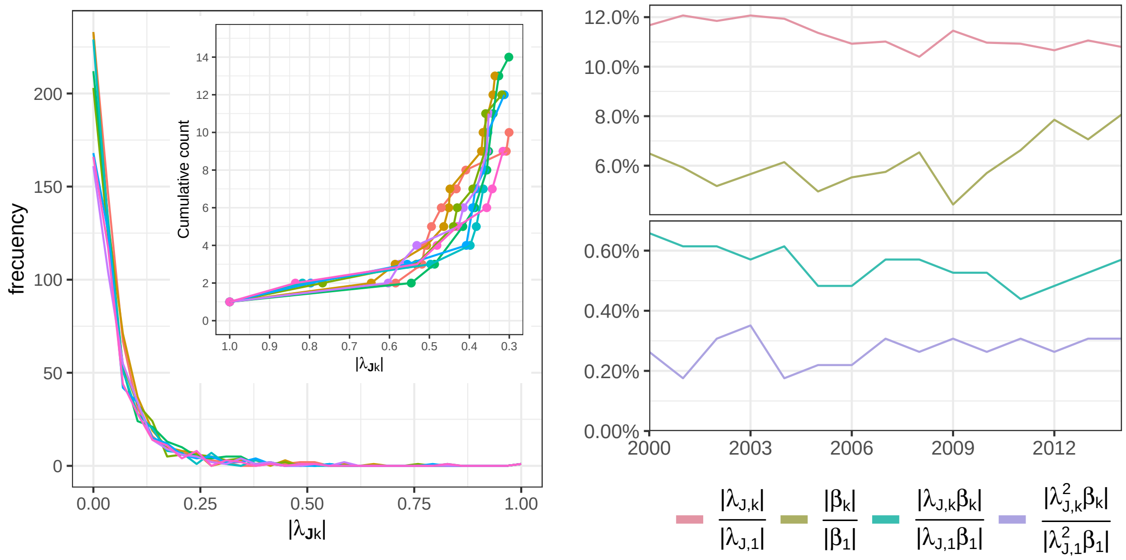
[4] See Bienenfeld (1988, p. 253), Iliadi et al. (2014), Mariolis (2021), Mariolis et al. (2021, chap. 2 and 3), Mariolis and Tsoulfidis (2009, 2011, 2014, 2016b,a, 2018), Shaikh (2016, pp. 410-2, 866) and Tsoulfidis (2021, chaps. 3 and 4; 2022). Schefold (2013a, p. 1173-4) considers the eigenvalues of matrix $\mathbf{A}$.

[5] $\mathbf{H} \equiv \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}$, where $\mathbf{A}$ is the input-coefficients matrix and $\lambda_{\mathbf{H}1}$ is the P-F eigenvalue of $\mathbf{H}$.

[6] In the extreme case of zero subdominant eigenvalues, $\mathbf{J}$ is a rank-one matrix. Theoretical models take as parameters the techniques of production whereas empirical computations use the available productive structure in monetary values. We will use the same mathematical symbols for the parameters in both models.

[7] E.g., Mariolis and Tsoulfidis (2011), Shaikh et al. (2020), and Torres-González and Yang (2019).

[8] Mariolis and Tsoulfidis (2014), Gurgul and Wojtowicz (2015), and Shaikh et al. (2020) show that the number of eigenvalues with a considerable magnitude tends to increase with the level of disaggregation.



(a) Eigenvalues' moduli for eight years, U.S. economy, 1977-2012.

(b) Percentage of subdominant variates above 0.25 for 43 countries.

Figure 1: $\lambda_{\mathbf{J}k}$ are the eigenvalues of matrix $\mathbf{J}$ and β_k are the eigelabours. $|\cdot|$ refers to modulus. Source: Plot (a), Torres-González (2020); Plot (b), authors' calculations based on the WIOD database, 2016 release.

curves and produce inflections. Thence, the explanation of the observed monotonicity/near monotonicity in price, capital intensities, and capital-output ratios curves requires *additional constraints on the productive structure.*

As stated in the epigraph from Sraffa, the curvatures depend on the variability in the proportions between means of production and labour —the proportions of the commodity inputs are just one side of the coin. Theoretical curves depend on the characteristics of both the input-coefficient matrix and its relationship with the labour-coefficient vector. Consistent with this focus, Torres-González (2020) conducts a spectral representation of the price model and shows that the curves depend on the eigenvalues $\lambda_{\mathbf{J}k}$, the eigenvectors, and what he calls the *eigenlabours* β_k, that is, the coefficients in the representation of the labour-coefficient vector in the space spanned by the left-hand eigenvectors of the input matrix $\mathbf{J}$. The eigenlabours β_k capture relevant information on the relationship between the input matrix and the labour vector. Section 2 shall show that we can get approximately linear (quadratic) price curves if the $\lambda_{\mathbf{J}k}\beta_k$'s ($\lambda^2_{\mathbf{J}k}\beta_k$'s) are sufficiently small. These theoretical results show that the first group of authors concentrates on a subset of the characteristics of the technique of production and leaves out other potentially relevant elements.

Torres-González (2017, 2020) studies the U.S. economy and shows that there is a statistical tendency of subdominant β_k towards zero. He conjectures that this result is connected to a statistical tendency towards (1) the proportionality of the labour vector and the P-F eigenvector and (2) uniform capital intensities.[9] Based on scalar

[9] In the extreme case of zero subdominant eigenvalues, labour vector and the P-F eigenvector

measures of the distance between the labour vector and the P-F eigenvector, Iliadi et al. (2014, p. 47) conclude that there are "considerable deviations" between them. In his explanation of the empirical near linearity of the wage curve, Schefold (2013a) also considers the relationship between the labour vector and the input matrix and proposes the hypothesis of zero average deviations between the labour vector and the P-F eigenvector —Section 5 shall provide empirical evidence for this hypothesis.

Sections 3 present evidence that for the WIOD database (i) the eigenvalues $\lambda_{\mathbf{J}k}$ by themselves do not fulfil the requirements for producing the curves consistent with the observed stylised facts and (ii) it is the *joint* action of the $\lambda_{\mathbf{J}k}$ and eigenlabours β_k which do meet these constraints. Plot (b) in Figure 1 show that less than .07% of the subdominant $|\lambda_{\mathbf{J}k}\beta_k|/\lambda_{\mathbf{J}1}\beta_1$ and $|\lambda_{\mathbf{J}k}^2\beta_k|/\lambda_{\mathbf{J}1}\beta_1$ are above 0.25, i.e., less that 1 observation per economy on average (c.f. 10%-12% of $\lambda_{\mathbf{J}k}/\lambda_{\mathbf{J}1}$). This result owes much to the magnitude of β_k, where only between 5%-8% of them are $|\beta_k|/\beta_1 \geq 0.25$ (between 2.6-4.2 observations per economy, on average). Section 4 provides robust evidence that the statistical regularities in the β_k are connected with a statistical tendency towards the proportionality between the labour vector and the P-F eigenvector. Section 5 develops some implications of the results for the literature and Section 6 concludes highlighting the lack of an *economic* explanation of the stylised facts of the literature.

2 Theory

2.1 The price system

Consider the standard Sraffian price of production model with no joint production, no fixed capital, and no land.[10] If wages are paid *ex-post*, then

$$\mathbf{p} = (1+r)\mathbf{pA} + w\mathbf{l}, \tag{1}$$

where $\mathbf{A} \equiv [a_{ij}] \geqq \mathbf{0}$ and $\mathbf{l} \equiv [l_j] > \mathbf{0}$ are the input-coefficients matrix and the labour-coefficients vector, and w and r are the wage and profit rates. It is assumed that $\mathbf{A}$ is productive, irreducible, primitive, and diagonalizable with n distinct eigenvalues, which are ordered as $1 > \lambda_{\mathbf{A}1} > |\lambda_{\mathbf{A}2}| \geq \cdots \geq |\lambda_{\mathbf{A}n}|$, where $\lambda_{\mathbf{A}1}$ is the P-F eigenvalue corresponding to the positive left- and right-hand eigenvectors $\mathbf{y}_1$ and $\mathbf{x}_1^T$. The labour vector is normalised such that the total quantity of labour is one, $\mathbf{lq}^T = 1$, where $\mathbf{q}^T \equiv [q_i]$ is the gross output.

Define the relative profit rate as $\rho \equiv \frac{r}{R}$, where R is the maximum rate of profit. We can express system (1) in terms of vertical integration as

are proportional and there are uniform capital intensities.

[10] We present the price system following closely Iliadi et al. (2014, sec. II) and Mariolis and Tsoulfidis (2018, sec. II). For a thorough exposition of the Sraffian price model see, inter alia, Pasinetti (1977, chap. 5).

$$\mathbf{p} = r\mathbf{pH} + w\mathbf{v} = \rho\mathbf{pJ} + w\mathbf{v}, \tag{2}$$

where $\mathbf{v} \equiv \mathbf{l}(\mathbf{I}-\mathbf{A})^{-1} > \mathbf{0}$ is the vector of vertically integrated (V.I.) labour coefficients, $\mathbf{J} \equiv \frac{1-\lambda_{\mathbf{A}1}}{\lambda_{\mathbf{A}1}}\mathbf{A}(\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{\lambda_{\mathbf{H}1}}\mathbf{H} > \mathbf{0}$ is the normalised matrix of V.I. input-coefficients, and $R = \frac{1}{\lambda_{\mathbf{H}1}}$.[11] Define the net output as $\mathbf{y}^T \equiv (\mathbf{I}-\mathbf{A})\mathbf{q}^T \equiv [y_i] \geq \mathbf{0}$. Hence, the labour normalisation can be expressed as $1 = \mathbf{lq}^T = \mathbf{vy}^T$.

By choosing the standard commodity as the numéraire, the wage-profit curve is $w = 1-\rho$ and price curves, i.e., prices $\mathbf{p} \equiv [p_j]$ as a function of ρ in the interval $0 \leq \rho < 1$, are

$$\begin{aligned}\mathbf{p}(\rho) &= (1-\rho)\mathbf{v}[\mathbf{I}-\rho\mathbf{J}]^{-1} \\ &= (1-\rho)[\mathbf{v} + \rho\mathbf{vJ} + \rho^2\mathbf{vJ}^2 + \cdots].\end{aligned} \tag{3}$$

2.2 Price-profit rate curves and their determinants

We now present an equivalent representation to equation (3) which will be helpful to determine the constraints in the techniques of production that shape the price curves:

$$\begin{aligned}\mathbf{p}(\rho) =& \mathbf{v} + \rho[\mathbf{vJ}-\mathbf{v}] + \rho^2[\mathbf{vJ}-\mathbf{v}]\mathbf{J} + \rho^3[\mathbf{vJ}-\mathbf{v}]\mathbf{J}^2 + \cdots \\ =& \mathbf{v} + \rho\mathbf{f} + \rho^2\mathbf{f}^2 + \rho^3\mathbf{f}^3 + \cdots \\ \mathbf{f}^q \equiv [f_j^q] \equiv& [\mathbf{vJ}-\mathbf{v}]\mathbf{J}^{q-1} = \mathbf{vJ}^{q-1}(\mathbf{J}-\mathbf{I}).\end{aligned} \tag{4}$$

Polynomial coefficients $\mathbf{f}^q$ depend exclusively on the means of production to labour proportions of all the industries of the economy. For $\mathbf{p}(\rho)$ to be, for example, approximately linear $\mathbf{p}(\rho) \approx \mathbf{v} + \rho\mathbf{f}$ it is required that $\rho^2\mathbf{f}^2 + \rho^3\mathbf{f}^3 + \cdots \approx \mathbf{0}$. Now, we follow Mariolis and Tsoulfidis (2018, sec. 2.c) and use the spectral representation of matrix $\mathbf{J}$. Under the assumption of distinct eigenvalues, matrix $\mathbf{J}$ can be expressed as:

$$\mathbf{J} = \sum\nolimits_{k=1}^{n} \frac{\lambda_{\mathbf{J},k}}{\mathbf{y}_k\mathbf{x}_k^T}\mathbf{x}_k^T\mathbf{y}_k = \mathbf{X}\mathbf{\Lambda}_{\mathbf{J}}\mathbf{X}^{-1}, \tag{5}$$

where $\mathbf{\Lambda}_{\mathbf{J}} = \mathrm{diag}\{\lambda_{\mathbf{J}1}, \ldots, \lambda_{\mathbf{J}n}\}$, $\lambda_{\mathbf{J}1} = 1$, and matrices $\mathbf{X}$ and $\mathbf{X}^{-1}$ contain as columns and rows respectively the right- and left-hand eigenvectors of matrix $\mathbf{J}$ when normalised such that $\mathbf{y}_k\mathbf{x}_k^T = 1$. Because $\mathbf{y}_s\mathbf{x}_k^T = 0$ for $k \neq s$ and $k, s = 1, \ldots, n$, for any normalisation of the eigenvectors, we have

$$\mathbf{vJ}^q = \sum\nolimits_{k=1}^{n} \frac{\lambda_{\mathbf{J}k}^q}{\mathbf{y}_k\mathbf{x}_k^T}\mathbf{vx}_k^T\mathbf{y}_k.$$

[11] Whereas matrices $\mathbf{A}$, $\mathbf{H}$ and $\mathbf{J}$ share the same eigenvectors $\mathbf{y}_k$ and $\mathbf{x}_k^T$, their eigenvalues differ but stand in a simple relationship: $\lambda_{\mathbf{H}k} = \lambda_{\mathbf{A}k}(1-\lambda_{\mathbf{A}k})^{-1}$ and $\lambda_{\mathbf{J}k} = (\lambda_{\mathbf{H}1})^{-1}\lambda_{\mathbf{H}k}$. n distinct eigenvalues imply in matrix $\mathbf{A}$ implies that it has n linearly independent eigenvectors $\mathbf{y}_k$ and $\mathbf{x}_k^T$.

Let us define coefficients $\beta_k \equiv \mathbf{v}\mathbf{x}_k^T$ and call them the "eigenlabours". Because the P-F eigenvector has positive entries $\mathbf{x}_1^T \in \mathbb{R}_+^n$ and the remaining $\mathbf{x}_k^T \in \mathbb{C}^n$, then $\beta_1 \in \mathbb{R}_+$ and $\beta_{k\geq 2} \in \mathbb{C}$. Hence $\mathbf{vJ}^q = \sum_{k=1}^n \left(\lambda_{\mathbf{J}k}^q / \mathbf{y}_k \mathbf{x}_k^T\right) \beta_k \mathbf{y}_k$ and $\lambda_{\mathbf{J}1} - 1 = 0$, so the polynomial coefficients in (4) are then:

$$\mathbf{f}^q = \sum\nolimits_{k=2}^n \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k \mathbf{x}_k^T} \lambda_{\mathbf{J}k}^{q-1} \beta_k \mathbf{y}_k. \tag{6}$$

The degree of the price curve polynomial (4), or the precision of any polynomial approximation, depends on the size of vectors $\mathbf{f}^q$, which can be measured by their vector norm $\|\mathbf{f}^q\|$, the bounds of which are

$$0 \leq \|\mathbf{f}^q\| \leq \sum\nolimits_{k=2}^n \left| \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k \mathbf{x}_k^T} \right| \left| \lambda_{\mathbf{J}k}^{q-1} \beta_k \right| \|\mathbf{y}_k\|, \tag{7}$$

where $|\cdot|$ indicates the absolute value. If we choose as the norm of vector $\mathbf{f}^q$ the maximum norm $\|\mathbf{f}^q\|_\infty := \max\left(|f_1^q|, \ldots, |f_n^q|\right) \equiv |f_{\max}^q|$, then, for $j = 1, \ldots, n$,

$$0 \leq |f_j^q| \leq |f_{\max}^q| \leq \sum\nolimits_{k=2}^n \left| \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k \mathbf{x}_k^T} \right| \left| \lambda_{\mathbf{J}k}^{q-1} \beta_k \right| |y_{k\,\max}|. \tag{8}$$

The different expressions of the polynomial coefficients (6)-(8) show clearly that the shapes of the price curves (4) depend on *the eigenvalues* $\lambda_{\mathbf{J}k}$, *the eigenlabours* β_k, and *the eigenvectors* $(\mathbf{y}_k, \mathbf{x}_k^T)$. It can be shown that price curves are constant $\mathbf{p}(\rho) = \mathbf{v}$ if and only if $\beta_2 = \cdots = \beta_n = 0$.[12] Linear price curves $\mathbf{p}(\rho) = \mathbf{v} + \rho\mathbf{f}$ are obtained if and only if $\lambda_{\mathbf{J}k}\beta_k = 0$ for $k \geq 2$, but without all $\beta_{k\geq 2} = 0$ (Torres-González, 2020, Section 2.4). Appendix A (Theorems 1 and 2) shows that although there cannot be exact quadratic or higher order polynomial price curves (only exact linear curves are possible), sufficient conditions for approximately linear (quadratic) curves are that $\lambda_{\mathbf{J}k}\beta_k \approx 0$ ($\lambda_{\mathbf{J}k}^2\beta_k \approx 0$), for $k = 2, \ldots, n$. Exact linear price curves do not depend on the scale of the eigenvectors $\mathbf{y}_k$, but their polynomial approximations do.

These constraints on price curves also constrain the curvature of industry's capital intensity $\mathbf{p}(\rho)\mathbf{H}_{(j)}/v_j$ and capital-output ratio $\mathbf{p}(\rho)\mathbf{H}_{(j)}/p(\rho)_j$ curves: Industries' V.I. capital value $\mathbf{p}(\rho)\mathbf{H}$, relative to the V.I. capital value of the standard industry $\lambda_{\mathbf{H},1}$, are

$$\mathbf{p}(\rho)\mathbf{J} = \mathbf{vJ} + \rho\mathbf{g}^1 + \rho^2\mathbf{g}^2 + \rho^3\mathbf{g}^3 + \cdots \tag{9}$$

$$\mathbf{g}^q \equiv [g_j^q] \equiv \mathbf{f}^q\mathbf{J} = \mathbf{f}^{q+1} = \sum\nolimits_{k=2}^n \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k \mathbf{x}_k^T} \lambda_{\mathbf{J}k}^q \beta_k \mathbf{y}_k. \tag{10}$$

Therefore, the constraints that produce approximate linear (quadratic) price curves also produce approximate constant (non-zero sloped linear) capital value curves: $\mathbf{f}^q = \mathbf{g}^{q-1} \approx \mathbf{0}$ for $q \geq 1$ (for $q \geq 2$) (see the corollaries to Theorems 1 and 2 in Appendix A).

[12] Because $\mathbf{A}$ and $\mathbf{J}$ are primitive, $\mathbf{f}^q$ cannot be the null vector due to $\lambda_{\mathbf{J}k} = 1 = \lambda_{\mathbf{J}1}$ for $k \geq 2$.

Expressions (6)-(8) and (10) show that deriving linear or approximate polynomial curves only through constraints on the eigenvalues is a particular case of more general conditions involving the eigenlabours and the eigenvectors. Suppose that $\lambda_{\mathbf{J},2} \approx \lambda_{\mathbf{J},3} \approx 0.5$, as in their empirical counterpart, and that the rest of the eigenvalues are equal to zero, $\lambda_{\mathbf{J},4} = \cdots = \lambda_{\mathbf{J},n} = 0$. Then, nearly linear price curves $\mathbf{p} \approx \mathbf{v} + \rho\mathbf{f}$ would be difficult to obtain unless $\beta_2, \beta_3 \approx 0$ and/or $\mathbf{y}_2, \mathbf{y}_3 \approx \mathbf{0}$ so that $0.5\beta_2\mathbf{y}_2 \approx 0.5\beta_3\mathbf{y}_3 \approx \mathbf{0}$.

2.3 Eigenlabours: their meaning and magnitude

The eigenlabours $\beta_k \equiv \mathbf{v}\mathbf{x}_k^T$ represent some aspects of the relationship between the labour vectors $(\mathbf{v}, \mathbf{l})$ and the input-coefficients matrices $(\mathbf{A}, \mathbf{J})$. On the one hand, given that $\mathbf{v} \equiv \mathbf{l}(\mathbf{I}-\mathbf{A})^{-1}$, the eigenlabours can be expressed as $\beta_k = \mathbf{l}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_k^T = \frac{\alpha_k}{1-\lambda_{\mathbf{A}k}}$, where $\alpha_k \equiv \mathbf{l}\mathbf{x}_k^T$. On the other hand, because the $\mathbf{x}_k$ are the eigenvectors of matrices $\mathbf{A}$ and $\mathbf{J}$, the eigenlabours $\mathbf{l}\mathbf{X} \equiv \boldsymbol{\alpha} \equiv [\alpha_k]$ and $\mathbf{v}\mathbf{X} \equiv \boldsymbol{\beta} \equiv [\beta_k]$ capture the linear relationship between the labour vectors $(\mathbf{v}, \mathbf{l})$ and the left-hand eigenvectors of the matrices:

$$\mathbf{l} = \boldsymbol{\alpha}\mathbf{X}^{-1} = \alpha_1\mathbf{y}_1 + \sum\nolimits_{k=2}^{n} \alpha_k\mathbf{y}_k \tag{11}$$

$$\mathbf{v} = \boldsymbol{\beta}\mathbf{X}^{-1} = \beta_1\mathbf{y}_1 + \sum\nolimits_{k=2}^{n} \beta_k\mathbf{y}_k = \frac{\alpha_1}{1-\lambda_{\mathbf{A}1}}\mathbf{y}_1 + \sum\nolimits_{k=2}^{n} \frac{\alpha_k}{1-\lambda_{\mathbf{A}k}}\mathbf{y}_k. \tag{12}$$

If $\mathbf{l}$ and $\mathbf{v}$ are proportional to the P-F eigenvector $\mathbf{y}_1$, then $\sum_{k=2}^{n} \alpha_k\mathbf{y}_k = \mathbf{0}$ and $\sum_{k=2}^{n} \beta_k\mathbf{y}_k = \mathbf{0}$. Because the eigenvectors $\mathbf{y}_k$ are linearly independent, the latter result can happen if and only if $\alpha_{k\geq 2} = \beta_{k\geq 2} = 0$. In this and only in this case, $\mathbf{l} \propto \mathbf{v}$, price curves are constant, and capital intensities are constant and uniform. Although the eigenlabours β_k are affected by the eigenvalues $\lambda_{\mathbf{A}k}$, the reduction of β_k due to small $\lambda_{\mathbf{A}k\geq 2}$ is limited: when $\lambda_{\mathbf{A}k} = 0$, then $\beta_k = \alpha_k$. Therefore, when compared with β_1, sufficiently small subdominant $\beta_{k\geq 2}$ can be postulated as a case of "strong proportionality" between $(\mathbf{v}, \mathbf{l})$ and $\mathbf{y}_1$.

The value of the eigenlabours depends on the scale of vectors $\mathbf{v}$ and $\mathbf{x}_k^T$ and the angle between them:[13]

$$\beta_k \equiv \mathbf{v}\mathbf{x}_k^T = \begin{cases} \|\mathbf{v}\|_2 \left\|\mathbf{x}_k^T\right\|_2 \cos\theta_k^{v,x_k}, & \text{if } \ \mathbf{x}_k^T \in \mathbb{R}^n \\ \|\mathbf{v}\|_2 \left\|\mathbf{x}_k^T\right\|_2 \cos\Theta_k^{v,x_k}, & \text{if } \ \mathbf{x}_k^T \in \mathbb{C}^n, \end{cases} \tag{13}$$

[13] In spite of the dependence of β_k to the scale of the eigenvectors $\mathbf{x}_k^T$, the terms $(\mathbf{y}_k\mathbf{x}_k^T)^{-1}\beta_k\mathbf{y}_k$ in (6) and (9) and $(\mathbf{y}_k\mathbf{x}_k^T)^{-1}\beta_k\|\mathbf{y}_k\|$ in (7) are invariant to the scale of the eigenvectors. Suppose $\mathbf{y}_k$ and $\mathbf{x}_k^T$ are arbitrarily scaled. Now, suppose $\mathbf{y}_k$ and $\mathbf{x}_k^T$ are re-scaled such that $\bar{\mathbf{y}}_k = \epsilon_k\mathbf{y}_k$ and $\bar{\mathbf{x}}_k^T = \varepsilon_k\mathbf{x}_k^T$, for $\epsilon_k, \varepsilon_k \neq 0$. Hence, $\bar{\beta}_k \equiv \mathbf{v}\bar{\mathbf{x}}_k^T$ and $\frac{\bar{\beta}_k}{\bar{\mathbf{y}}_k\bar{\mathbf{x}}_k^T}\bar{\mathbf{y}}_k = \frac{\mathbf{v}\bar{\mathbf{x}}_k^T}{\bar{\mathbf{y}}_k\bar{\mathbf{x}}_k^T}\bar{\mathbf{y}}_k = \frac{\mathbf{v}(\varepsilon_k\mathbf{x}_k^T)}{(\epsilon_k\mathbf{y}_k)(\varepsilon_k\mathbf{x}_k^T)}(\epsilon_k\mathbf{y}_k) = \frac{\beta_k}{\mathbf{y}_k\mathbf{x}_k^T}\mathbf{y}_k$. If the normalisation $\varepsilon_k = \frac{1}{\beta_k}$ is selected such that $\bar{\beta}_k = 1$, then $\frac{1}{\mathbf{y}_k\bar{\mathbf{x}}_k^T}\mathbf{y}_k = \frac{1}{\mathbf{y}_k\mathbf{x}_k^T}\frac{1}{\varepsilon_k}\mathbf{y}_k = \frac{\beta_k}{\mathbf{y}_k\mathbf{x}_k^T}\mathbf{y}_k$. This shows that it is not possible to discard the influence of the relationship between the labour vector $\mathbf{v}$ and the eigenvectors $\mathbf{x}_k^T$ by choosing a special normalisation of the latter.

where $\|\cdot\|_2$ is the Euclidean vector norm and θ_k^{v,x_k} and Θ_k^{v,x_k} are the real and complex-valued angles between vectors $\mathbf{v}$ and $\mathbf{x}_k^T$. The angles are bounded by $-1 \leq \cos\theta_k^{v,x_k} \leq 1$ and $0 \leq |\cos\theta_k^{v,x_k}|, |\cos\Theta_k^{v,x_k}| \leq 1$ (see Scharnhorst, 2001, p. 96-7). The size of $\mathbf{x}_k^T$ can be set arbitrarily, so if by setting $\|\mathbf{x}_k\|_2 = 1$ for $k = 1, \ldots, n$, then the eigenlabours in terms of the $\|\mathbf{v}\|_2$ depend only on the angle between the vectors and

$$\frac{\beta_k}{\beta_1} = \frac{\mathbf{v}\mathbf{x}_k^T}{\mathbf{v}\mathbf{x}_1^T} = \begin{cases} \frac{\cos\theta_k^{v,x_k}}{\cos\theta_1^{v,x_k}} & \text{if } \mathbf{x}_{k\neq 1}^T \in \mathbb{R}^n \\ \frac{\cos\Theta_k^{v,x_k}}{\cos\theta_1^{v,x_k}} & \text{if } \mathbf{x}_{k\neq 1}^T \in \mathbb{C}^n. \end{cases} \tag{14}$$

Regarding $\|\mathbf{v}\|_2$, the labour normalisation $\mathbf{l}\mathbf{q}^T = \mathbf{v}\mathbf{y}^T = 1$ implies that $\|\mathbf{v}\|_2 = 1/\left\|\mathbf{y}^T\right\|_2 \cos\theta^{v,y}$. Although $\left\|\mathbf{y}^T\right\|_2$ and $\cos\theta^{v,y}$ vary across economies, they are the same for all β_k.

Let us set $\|\mathbf{x}_k\|_2 = 1$ for $k = 1, \ldots, n$ and collect them as the columns of matrix $\mathbf{X}$, as in (5). If the left-hand eigenvectors are obtained as the rows of matrix $\mathbf{X}^{-1}$, then $\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$ implies that $\mathbf{y}_k\mathbf{x}_k^T = 1$ and[14]

$$\|\mathbf{y}_k\|_2 = \begin{cases} \frac{1}{\cos\theta_k^{y_k,x_k}}, & \text{if } \mathbf{x}_k^T \in \mathbb{R}^n \\ \frac{1}{\cos\Theta_k^{y_k,x_k}}, & \text{if } \mathbf{x}_k^T \in \mathbb{C}^n \end{cases} \quad \text{and}$$

$$0 \leq \frac{\|\mathbf{f}^q\|_\infty}{\|\mathbf{v}\|_2} \leq \frac{\|\mathbf{f}^q\|_2}{\|\mathbf{v}\|_2} \leq \begin{cases} \sum_{k=2}^n |\lambda_{\mathbf{J}k} - 1| \left|\lambda_{\mathbf{J}k}^{q-1}\right| \left|\frac{\cos\theta_k^{v,x_k}}{\cos\theta_k^{y_k,x_k}}\right|, & \text{if } \mathbf{x}_k^T \in \mathbb{R}^n \\ \sum_{k=2}^n |\lambda_{\mathbf{J}k} - 1| \left|\lambda_{\mathbf{J}k}^{q-1}\right| \left|\frac{\cos\Theta_k^{v,x_k}}{\cos\Theta_k^{y_k,x_k}}\right|, & \text{if } \mathbf{x}_k^T \in \mathbb{C}^n. \end{cases}$$

That is, the magnitude of the polynomial coefficients $\mathbf{f}^q$ relative to $\|\mathbf{v}\|_2$ does not depend on the scale of any vector —only on angles and the eigenvalues.

3 Eigenvalues and eigenlabours in the economies of the World Input-Output Database

Section 2 showed that the polynomial order of the price (4) and capital (9) curves depends on coefficients $\mathbf{f}^q$ and $\mathbf{g}^q$ in (6) and (10). Under the spectral representation of the price system, these coefficients depend on the the subdominant (i.e, for $k = 2, \ldots, n$) eigenvalues $\lambda_{\mathbf{J}k}^{q-1}$, eigenlabours β_k, and eigenvectors $\mathbf{y}_k$. A set of constraints on the subdominant $\lambda_{\mathbf{J}k}^{q_0}\beta_k$ where introduced to produce polynomial approximations of order q_0 to price and capital curves (Theorem 2 and Corollary 2, Appendix A). Now, it is a stylised fact that the curves computed with information from input-output accounts of actual economies are well approximated by linear or quadratic functions. Most explanations of these regularities postulate the hypothesis of sufficiently small subdominant eigenvalues $\lambda_{\mathbf{J}k\geq 2} \approx 0$.

[14] The constraint $\mathbf{y}_k\mathbf{x}_k^T = 1$ forces their angle to have a zero imaginary part when $\mathbf{y}_k, \mathbf{x}_k^T \in \mathbb{C}^n$. So, $\cos\Theta_k^{y,x_k} = |\cos\Theta_k^{y,x_k}|$.

Sections 3.2-3.3 and the robustness appendix D take the productive structures of *all the economies* in the WIOD database and provide robust evidence on the existence the following stylised facts (*SF*):

SF1 there is a statistical tendency towards $\lambda_{\mathbf{J}k}\beta_k = 0$ and $\lambda^2_{\mathbf{J}k}\beta_k = 0$ for $k = 2, \ldots, n$.

SF2 in spite of the clustering of subdominant eigenvalues $\lambda_{\mathbf{J}k}$ around small values, there is an important number of observations with a considerable magnitude which makes them unable to produce *by themselves and without any reference to the eigenlabours* β_k the statistical tendency in *SF1*.

SF3 there is a statistical tendency towards $\beta_k = 0$ for $k = 2, \ldots, n$.

Section 5 develops the implications of these and other results. Our empirical analysis abstracts from the characteristics of the scale of the eigenvectors $\mathbf{y}_k$ and the angle formed between $\mathbf{y}_k$ and $\mathbf{x}_k^T$.

3.1 The database with the productive structures

Here we provide a brief description of the main characteristics of the database. Its full description and the appropriate transformations of the data can be found in Appendix B.

Sample. The calculations presented draw from the World Input-Output Database (WIOD) (Timmer et al., 2016), from which we construct a database with the productive structures of 645 economies belonging to 43 countries for a period of 15 years (2000-2014). This sample contains high- and middle-income countries which account for 86% of the world economy in 2016. Each economy shares a homogeneous set of 54 industries.

We consider the subset of eight economies for 2011 used in Mariolis and Tsoulfidis (2018): Australia (AUS), Brazil (BRA), China (CHN), France (FRA), Germany (DEU), India (IND), Japan (JPN), and the United States (USA) –we refer to this sample by "MT2011". When expanding the MT2011 sample to encompass the whole period from 2000 to 2014, which we call "MT15". Whenever we aggregate through the 43 countries for the year 2011, we will refer to it as "WIOD2011". If the aggregation involves all countries and years, we will use the short-hand "WIOD".

Data construction. Leontief's technical coefficient matrices are constructed aggregating domestic and imported intermediate inputs. In accordance with the relevant literature (e.g. Mariolis and Tsoulfidis 2016b, p. 222; Shaikh 1998, p. 98), Section 3.2 constructs the skill-adjusted labour-coefficient vector using the compensation received by the persons engaged in production (employees + self-employed) divided by the economy-wide average wage rate. In Section 3.3 and appendices D.2-D.3 we test the robustness of the results to changes in the labour measurement unit and use a conventional labour vector comprising only persons engaged.

Normalisations. Right-hand eigenvectors will be normalised as $\|\mathbf{x}_k\|_2 = 1$, for $k = 1, \ldots, n$. Left-hand eigenvectors $\mathbf{y}_k$ are obtained as the rows of matrix $\mathbf{X}^{-1}$, where $\mathbf{X}$ has as columns the eigenvectors $\mathbf{x}_k$ (as in (5)). Finally, we normalise the labour vector so that $\mathbf{lq}^T = L = 1$, as in Section 2.1.

3.2 The subdominant eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and the $\lambda_{\mathbf{J}k}\beta_k$, and $\lambda^2_{\mathbf{J}k}\beta_k$

Clustering of the subdominant observations. We begin presenting the evidence for stylised facts *SF1*, *SF2*, and *SF3* by observing the *empirical distribution* of subdominant eigenvalues ($\lambda_{\mathbf{J}k}$), eigenlabours (β_k), and their products ($\beta_k\lambda_{\mathbf{J}k}$, $\beta_k\lambda^2_{\mathbf{J}k}$) in Figure 2. Because these variates are complex numbers we represent their frequencies using two-dimensional (2-D) histogram plots, where the X- and Y-axis represents the real and imaginary component, respectively. Each squared bin contains the number of observations within that real- and imaginary-component value range. The colour of the square (in log base 10) represents the number of observations with that complex value —the darker the colour the more populated that value-range is. The first eight rows of plots correspond to the MT2011 sample. The last two rows pool the observations of all countries for year 2011 (WIOD2011) and for all years (WIOD).

The first characteristic of the 2D histograms is that there is a strong concentration of observations around zero across parameters, countries and time periods. Most of the sample is located in a relatively small neighbourhood centred around zero. The dark blued squares, which denote where most observations are located, are around the origin. Squares outside the neighbourhood at the origin are mostly populated with green, yellow, and grey colours, which denote small and null observations. Hence, only a small proportion of observations lay outside the neighbourhood around zero. The patterns for the eight countries of the MT2011 sample verify for each of the 43 countries in the WIOD (see Appendix C.1, figures C.1-C.3). The second characteristic is that the closer we get to the origin from any direction the more populated the squares become. That is, the clustering around zero intensifies as we approach the origin. This is more evident for the highly populated samples MT2011 and WIOD.

In spite that all variates fit this general description, there are differences which are of paramount importance for the characterisation of price and capital value curves. While the pattern is particularly strong in the case of β_k, $\lambda_{\mathbf{J}k}\beta_k$, and $\lambda^2_{\mathbf{J}k}\beta_k$,

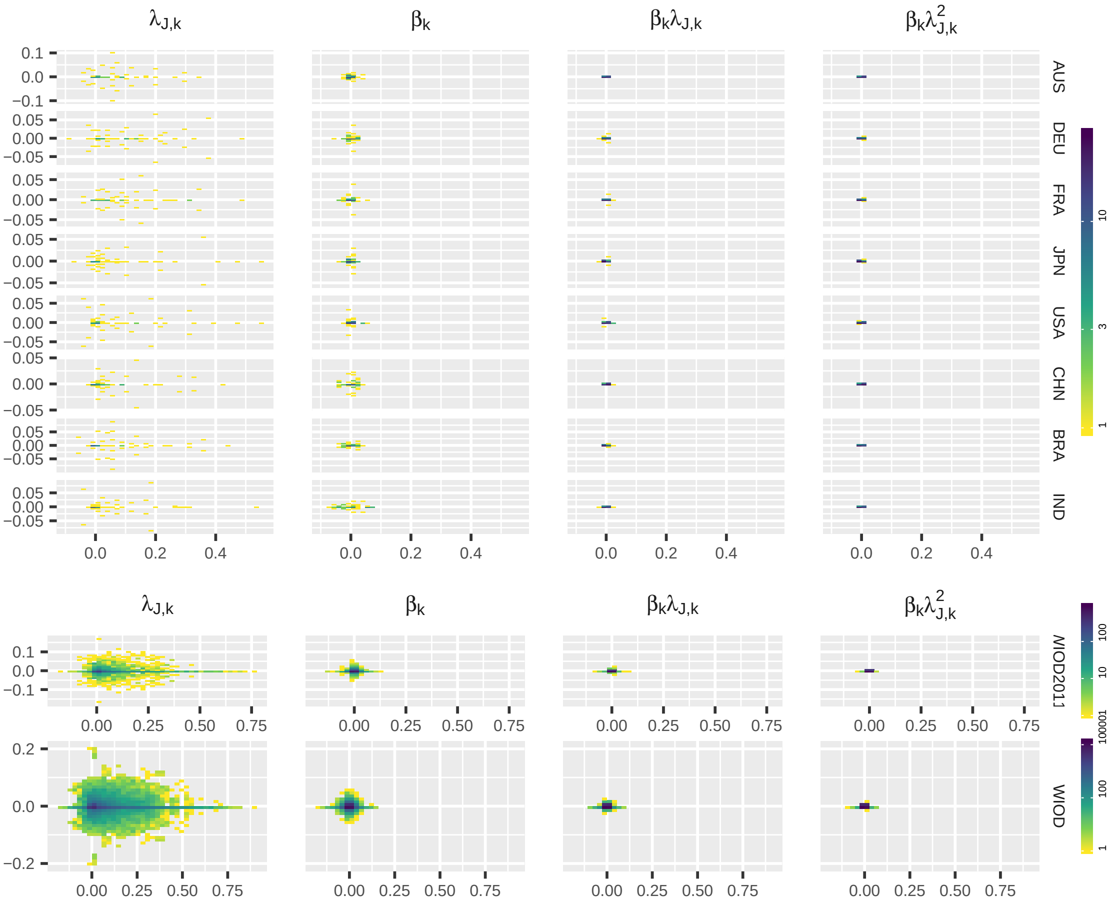


Figure 2: 2-D histogram of subdominant ($k \geq 2$) eigenvalues $\lambda_{\mathbf{J}_k}$, eigenlabours β_k, and their products ($\lambda_{\mathbf{J}k}\beta_k, \lambda^2_{\mathbf{J}k}\beta_k$) for different samples. X- and Y-axis represent the real and imaginary parts of the variate, respectively. Source: authors' calculations based on the WIOD database, 2016 release.

where almost their entire mass locates in a tinny neighbourhood around zero, the same is not true for the eigenvalues $\lambda_{\mathbf{J}k}$. Eigenvalues display more variability so the neighbourhood which captures most of the observations is considerably wider than that of the β_k, $\lambda_{\mathbf{J}k}\beta_k$, and $\lambda^2_{\mathbf{J}k}\beta_k$. In addition, there are many more $\lambda_{\mathbf{J}k}$ outside this neighbourhood and with considerable magnitude —e.g., see the $\lambda_{\mathbf{J}k}$ along the positive segment of the X-axis. Therefore, the higher variability in the $\lambda_{\mathbf{J}k}$ coupled with an important number of observations with a considerable magnitude imply that $\lambda_{\mathbf{J}k}$ cannot be responsible of the statistical tendency towards zero of the $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ without any reference to the eigenlabours β_k. The eigenlabours β_k contribute importantly to *SF1*.

The moduli of the largest observations. We now turn our attention to the characterisation of the observations *outside the neighbourhood around zero*. For this, we obtain the *modulus* or magnitude of the parameters $|\lambda_{\mathbf{J}k}|, |\beta_k|, |\lambda_{\mathbf{J}k}\beta_k|, |\lambda^2_{\mathbf{J}k}\beta_k|$ and consider some descriptive statistics in Figure 3 and Table 1. Figure 3 present the boxplot of *the subdominant* observations for the individual countries in the MT2011 sample (the eight rows in the first horizontal block), and for the aggregate

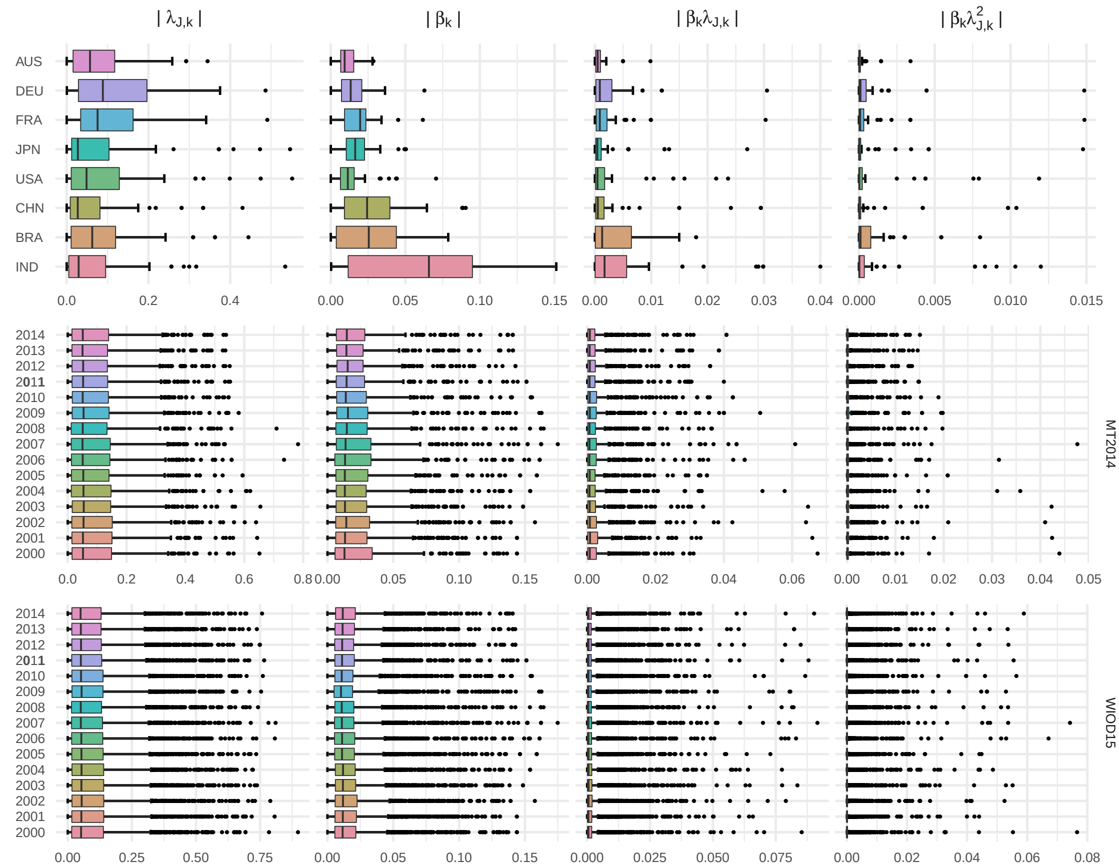


Figure 3: Boxplots of the moduli of the subdominant ($k \geq 2$) eigenvalues $|\lambda_{\mathbf{J}k}|$, eigenlabours $|\beta_k|$, and their products ($|\lambda_{\mathbf{J}k}\beta_k|, |\lambda^2_{\mathbf{J}k}\beta_k|$) for different samples. Source: authors' calculations based on the WIOD database, 2016 release.

of countries for each year of the MT15 and WIOD samples (the 15 rows in the second and third horizontal block).[15] Table 1 presents the median, the interquartile range (IQR), and the right whisker (RW) of the variates for each of the 43 countries in 2011. In addition, for each variate $\delta_k \in (\lambda_{\mathbf{J}k}, \beta_k, \lambda_{\mathbf{J}k}\beta_k, \lambda^2_{\mathbf{J}k}\beta_k)$ the fourth column presents the "dominant" δ_1 observation, the fifth column the $\frac{\max|\delta_{k\geq 2}|}{\delta_1}$, that is, the ratio of the maximum subdominant magnitude and the dominant, and the sixth column $\frac{\text{sub}|\delta_{k\geq 2}|}{\delta_1}$, where $\text{sub}|\delta_{k\geq 2}|$ denotes the second maximum subdominant magnitude. Hence, columns 5 and 6 tell how much are the largest $\delta_{k\geq 2}$ in terms of δ_1, a crucial statistic for the characterisation of the price and capital value curves.[16]

Based on Figure 3 and Table 1, we can characterise the relatively small propor-

[15] A boxplot summarises graphically the shape of a distribution by drawing the median (second quartile, Q2) within the box set by the first quartile (Q1) and the third quartile (Q3), i.e., the interquartile range (IQR). Additionally, the whiskers (the vertical bars) pinpoint the extreme limits of the distribution. Any data point outside these boundaries can be classified as an outlier, and it is indicated as a fully drawn point. The left whisker (LW) corresponds to the maximum of Q1 - 1.5IQR or the minimum value whereas the right whisker (RW) corresponds to the minimum of Q3 + 1.5IQR or the maximum value.

[16] In general $\beta_1 \not\geq |\beta_{k\geq 2}|$ for $k \geq 2$ as we have with eigenvalues $\lambda_{\mathbf{J}1} > |\lambda_{\mathbf{J}k}|$. See paragraph "speed of convergence".

tion of observations β_k, $\lambda_{\mathbf{J}k}\beta_k$, and $\lambda^2_{\mathbf{J}k}\beta_k$ outside the vicinity around zero found in Figure 2 as follows: (1) there are only few observations, (2) these can be considered outliers, and (3) their magnitude is pretty small. This is not the case for the subdominant eigenvalues $\lambda_{\mathbf{J}k}$: there are many observations outside the neighbourhood around zero, these observations are not outliers in all cases, and their magnitude is considerably large.

Consider the variates $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ for the MT2011 sample. Figure 3 shows that boxplot's right whisker, which correspond to the non-outlier maximum value, is at most 0.005 (for IND and BRA, $|\lambda_{\mathbf{J}k}\beta_k|$). Table 1 show that for the 43 countries in 2011 the average values of the RW (IQR) for $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ are 0.003 (0.0001) and 0.0000 (0.0000), respectively. DEU and IND have 4 outliers for $|\lambda_{\mathbf{J}k}\beta_k|$. The rest of the samples for both $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ have similar number of outliers. However, these 3, 4, 5 outliers are a small proportion of the sample for each country-year and, more importantly, their magnitude is considerably small: their maximum value is 0.03 (DEU and FRA, $|\lambda_{\mathbf{J}k}\beta_k|$). The boxplots for the variates $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ for each country-year (figures C.4-C.9) and the descriptive statistics (tables C.1-C.2) for years 2000 and 2014 in Appendix C.2 tell the same story. The boxplot for the samples MT15 and WIOD in Figure 3 summarise this characterisation: the right whiskers are indistinguishable from zero and the maximum outliers for $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ in the 645 economies are around 0.1 and 0.0875, respectively.

These regularities of the largest subdominant $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ relate quite differently with the eigenlabours $|\beta_k|$ and the eigenvalues $|\lambda_{\mathbf{J}k}|$. Whereas the $|\beta_{k\geq 2}|$ have a similar behaviour, the $|\lambda_{\mathbf{J}k\geq 2}|$ differ considerably. There are many $|\lambda_{\mathbf{J}k\geq 2}|$ and with a considerable magnitude compared with the $|\beta_{k\geq 2}|$. Consider the MT2011 sample. The RW of the $|\beta_{k\geq 2}|$ locate between 0.01 (USA) and 0.08 (IND) whereas those for the $|\lambda_{\mathbf{J}k\geq 2}|$ locate between 0.18 and 0.37. For the 43 countries in 2011, the average RW for $|\beta_{k\geq 2}|$ is 0.0033 *vis-a-vis* 0.312 for $|\lambda_{\mathbf{J}k\geq 2}|$ (see Table 1). When considering the MT15 and WIOD samples, the RW for the $|\lambda_{\mathbf{J}k\geq 2}|$ seem to be no less that 0.3 *vis-a-vis* the $|\beta_{k\geq 2}|$ which is no greater than 0.05. In 2011, JAP and the USA have three outliers $|\lambda_{\mathbf{J}k\geq 2}| \geq 0.4$. Table 1 show that for the 43 countries in 2011 the average $|\lambda_{\mathbf{J}2}|$ and $|\lambda_{\mathbf{J}3}|$ equal 0.53 and 0.4, respectively. For the WIOD sample, the outliers of $|\lambda_{\mathbf{J}k\geq 2}|$ reach magnitudes between 0.5 and 0.75 whereas the maximal $|\beta_{k\geq 2}|$ seem to be less than 0.17.

Speed of convergence. We now compute the moduli of the eigenlabours $|\beta_k|$ and the eigenvalues $|\lambda_{\mathbf{J}k}|$ in terms of their "dominant" observation $|\beta_1|$ and $\lambda_{\mathbf{J}1}$ in order to *compare their speed of convergence towards zero* and provide further evidence in favour of *SF2-SF3*. We use the rank-plot of the normalised moduli of the eigenvalues $\lambda_{\mathbf{J}k}/\lambda_{\mathbf{J}1} = \lambda_{\mathbf{J}k}$ and eigenlabours $|\beta_k|/\beta_1$ for the eight countries in the MT15 sample in Figure 4 and the fourth and fifth columns for $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ in Table 1. Appendix C

Country	$\lvert\lambda_{\mathbf{J}k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda_{\mathbf{J}k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda^2_{\mathbf{J}k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$
AUS	0.057	0.101	0.270	1	0.345	0.292	0.008	0.009	0.025	0.198	0.170	0.131	0.000	0.001	0.002	0.198	0.044	0.038	0	0.000	0.000	0.198	0.015	0.011
AUT	0.079	0.182	0.490	1	0.581	0.535	0.008	0.008	0.026	0.168	0.439	0.181	0.001	0.003	0.007	0.168	0.235	0.102	0	0.001	0.002	0.168	0.126	0.059
BEL	0.069	0.157	0.419	1	0.439	0.398	0.007	0.010	0.030	0.227	0.209	0.181	0.001	0.001	0.003	0.227	0.065	0.064	0	0.000	0.000	0.227	0.023	0.022
BGR	0.048	0.123	0.322	1	0.490	0.303	0.006	0.007	0.022	0.149	0.196	0.168	0.000	0.001	0.002	0.149	0.059	0.031	0	0.000	0.000	0.149	0.018	0.012
BRA	0.062	0.120	0.312	1	0.445	0.363	0.011	0.024	0.063	0.157	0.248	0.212	0.001	0.002	0.006	0.157	0.110	0.077	0	0.000	0.001	0.157	0.049	0.028
CAN	0.063	0.152	0.409	1	0.552	0.445	0.009	0.014	0.039	0.184	0.171	0.151	0.000	0.001	0.003	0.184	0.094	0.044	0	0.000	0.001	0.184	0.052	0.016
CHE	0.050	0.137	0.352	1	0.630	0.559	0.014	0.015	0.043	0.125	1.120	1.050	0.001	0.001	0.003	0.125	0.706	0.587	0	0.000	0.001	0.125	0.445	0.328
CHN	0.028	0.081	0.213	1	0.430	0.334	0.017	0.018	0.051	0.190	0.249	0.246	0.000	0.001	0.002	0.190	0.102	0.082	0	0.000	0.000	0.190	0.044	0.027
CYP	0.052	0.115	0.309	1	0.601	0.379	0.015	0.009	0.034	0.123	0.307	0.263	0.001	0.002	0.005	0.123	0.098	0.096	0	0.000	0.001	0.123	0.059	0.030
CZE	0.065	0.148	0.400	1	0.596	0.396	0.007	0.007	0.021	0.223	0.253	0.163	0.000	0.002	0.005	0.223	0.151	0.064	0	0.000	0.001	0.223	0.090	0.026
DEU	0.092	0.181	0.481	1	0.487	0.375	0.011	0.013	0.038	0.200	0.308	0.178	0.001	0.002	0.004	0.200	0.150	0.067	0	0.000	0.000	0.200	0.073	0.025
DNK	0.067	0.091	0.265	1	0.601	0.491	0.006	0.008	0.024	0.202	0.380	0.110	0.000	0.001	0.002	0.202	0.186	0.066	0	0.000	0.000	0.202	0.092	0.040
ESP	0.071	0.113	0.316	1	0.516	0.301	0.009	0.009	0.027	0.180	0.377	0.325	0.001	0.002	0.004	0.180	0.112	0.098	0	0.000	0.001	0.180	0.033	0.030
EST	0.057	0.136	0.364	1	0.608	0.460	0.006	0.004	0.014	0.170	0.372	0.367	0.000	0.001	0.003	0.170	0.226	0.169	0	0.000	0.000	0.170	0.137	0.078
FIN	0.070	0.119	0.320	1	0.376	0.304	0.007	0.003	0.014	0.204	0.161	0.107	0.000	0.001	0.002	0.204	0.049	0.016	0	0.000	0.000	0.204	0.015	0.006
FRA	0.081	0.127	0.354	1	0.491	0.341	0.012	0.012	0.035	0.187	0.324	0.216	0.001	0.001	0.003	0.187	0.159	0.073	0	0.000	0.001	0.187	0.078	0.025
GBR	0.088	0.146	0.402	1	0.472	0.394	0.007	0.013	0.034	0.162	0.326	0.296	0.001	0.001	0.004	0.162	0.126	0.097	0	0.000	0.001	0.162	0.049	0.033
GRC	0.049	0.091	0.239	1	0.580	0.467	0.011	0.014	0.042	0.115	0.640	0.374	0.001	0.001	0.003	0.115	0.371	0.051	0	0.000	0.000	0.115	0.215	0.015
HRV	0.061	0.102	0.282	1	0.540	0.355	0.006	0.005	0.017	0.103	0.274	0.201	0.000	0.001	0.003	0.103	0.105	0.071	0	0.000	0.000	0.103	0.057	0.025
HUN	0.047	0.088	0.243	1	0.650	0.356	0.008	0.007	0.022	0.170	0.594	0.293	0.000	0.001	0.002	0.170	0.386	0.072	0	0.000	0.000	0.170	0.251	0.018
IDN	0.046	0.120	0.311	1	0.766	0.590	0.013	0.016	0.044	0.119	0.863	0.510	0.000	0.001	0.002	0.119	0.510	0.187	0	0.000	0.000	0.119	0.301	0.086
IND	0.033	0.110	0.281	1	0.535	0.317	0.029	0.039	0.108	0.136	0.574	0.515	0.001	0.002	0.006	0.136	0.172	0.159	0	0.000	0.001	0.136	0.052	0.050
IRL	0.039	0.076	0.203	1	0.444	0.255	0.010	0.009	0.030	0.053	2.270	0.600	0.000	0.001	0.002	0.053	1.008	0.153	0	0.000	0.000	0.053	0.447	0.039
ITA	0.050	0.130	0.337	1	0.425	0.380	0.007	0.008	0.024	0.191	0.212	0.184	0.000	0.001	0.003	0.191	0.060	0.057	0	0.000	0.000	0.191	0.018	0.017
JPN	0.027	0.094	0.246	1	0.547	0.473	0.011	0.008	0.026	0.126	0.372	0.245	0.000	0.001	0.003	0.126	0.204	0.091	0	0.000	0.000	0.126	0.111	0.034
KOR	0.037	0.107	0.278	1	0.386	0.328	0.007	0.008	0.023	0.153	0.250	0.175	0.000	0.001	0.001	0.153	0.041	0.039	0	0.000	0.000	0.153	0.013	0.010
LTU	0.091	0.142	0.398	1	0.497	0.396	0.008	0.006	0.020	0.113	0.357	0.154	0.001	0.001	0.003	0.113	0.177	0.045	0	0.000	0.000	0.113	0.088	0.015
LUX	0.016	0.067	0.170	1	0.399	0.232	0.008	0.011	0.030	0.052	1.875	0.993	0.000	0.000	0.001	0.052	0.749	0.230	0	0.000	0.000	0.052	0.299	0.053
LVA	0.046	0.111	0.298	1	0.543	0.385	0.008	0.006	0.020	0.109	0.408	0.280	0.000	0.001	0.003	0.109	0.125	0.075	0	0.000	0.000	0.109	0.041	0.038
MEX	0.030	0.074	0.194	1	0.708	0.454	0.009	0.008	0.024	0.068	0.492	0.488	0.000	0.000	0.001	0.068	0.346	0.096	0	0.000	0.000	0.068	0.245	0.030
MLT	0.030	0.105	0.272	1	0.527	0.432	0.015	0.015	0.048	0.146	0.269	0.227	0.000	0.001	0.003	0.146	0.054	0.041	0	0.000	0.000	0.146	0.028	0.018
NLD	0.082	0.136	0.375	1	0.548	0.429	0.005	0.006	0.019	0.180	0.378	0.211	0.000	0.001	0.002	0.180	0.207	0.091	0	0.000	0.000	0.180	0.114	0.039
NOR	0.048	0.100	0.268	1	0.565	0.460	0.006	0.006	0.017	0.182	0.215	0.158	0.000	0.001	0.002	0.182	0.122	0.073	0	0.000	0.000	0.182	0.069	0.034
POL	0.061	0.114	0.314	1	0.424	0.358	0.011	0.009	0.032	0.188	0.159	0.159	0.001	0.002	0.005	0.188	0.067	0.048	0	0.000	0.001	0.188	0.029	0.016
PRT	0.073	0.182	0.484	1	0.579	0.460	0.009	0.009	0.029	0.145	0.285	0.240	0.001	0.002	0.006	0.145	0.165	0.111	0	0.000	0.001	0.145	0.096	0.051
ROU	0.022	0.051	0.135	1	0.258	0.172	0.007	0.006	0.019	0.164	0.208	0.104	0.000	0.000	0.001	0.164	0.054	0.006	0	0.000	0.000	0.164	0.014	0.001
RUS	0.000	0.059	0.147	1	0.485	0.312	0.002	0.018	0.045	0.194	0.377	0.209	0.000	0.001	0.002	0.194	0.183	0.061	0	0.000	0.000	0.194	0.089	0.019
SVK	0.060	0.127	0.336	1	0.703	0.524	0.011	0.008	0.025	0.150	0.555	0.362	0.000	0.001	0.003	0.150	0.390	0.167	0	0.000	0.001	0.150	0.274	0.087
SVN	0.089	0.169	0.457	1	0.690	0.426	0.011	0.009	0.028	0.212	0.137	0.129	0.001	0.002	0.005	0.212	0.095	0.032	0	0.000	0.001	0.212	0.066	0.012
SWE	0.051	0.096	0.255	1	0.410	0.376	0.007	0.007	0.021	0.213	0.202	0.095	0.000	0.001	0.003	0.213	0.076	0.027	0	0.000	0.000	0.213	0.028	0.011
TUR	0.026	0.121	0.309	1	0.662	0.525	0.014	0.021	0.063	0.137	0.465	0.388	0.001	0.002	0.005	0.137	0.212	0.113	0	0.000	0.000	0.137	0.140	0.047
TWN	0.035	0.103	0.274	1	0.638	0.371	0.014	0.023	0.064	0.053	1.671	1.438	0.001	0.001	0.004	0.053	0.582	0.221	0	0.000	0.001	0.053	0.202	0.034
USA	0.049	0.120	0.312	1	0.552	0.475	0.006	0.007	0.021	0.154	0.378	0.280	0.000	0.001	0.003	0.154	0.144	0.133	0	0.000	0.000	0.154	0.079	0.063
Mean	0.053	0.117	0.312	1	0.528	0.395	0.010	0.011	0.033	0.155	0.373	0.257	0.000	0.001	0.003	0.155	0.175	0.087	0	0.000	0.000	0.155	0.092	0.035
SD	0.021	0.031	0.086	0	0.107	0.089	0.004	0.007	0.017	0.045	0.455	0.269	0.000	0.001	0.002	0.045	0.211	0.092	0	0.000	0.000	0.045	0.110	0.050

Table 1: Descriptive statistics of the moduli of the eigenvalues $|\lambda_{\mathbf{J}k}|$, eigenlabours $|\beta_k|$, and their products $(|\lambda_{\mathbf{J}k}\beta_k|, |\lambda^2_{\mathbf{J}k}\beta_k|)$ for 2011. The columns of each variate reports the median, the interquartile range (IQR), the right whisker (RW), the "dominant" observation $k = 1$, the maximum observation for $k \geq 2$, and the second maximum observation for $k \geq 2$ (the last two normalised by the corresponding $k = 1$). Grey-coloured rows refer to cases where $\beta_{k\geq2} \geq \beta_1$. Source: authors' calculations based on the WIOD database, 2016 release.

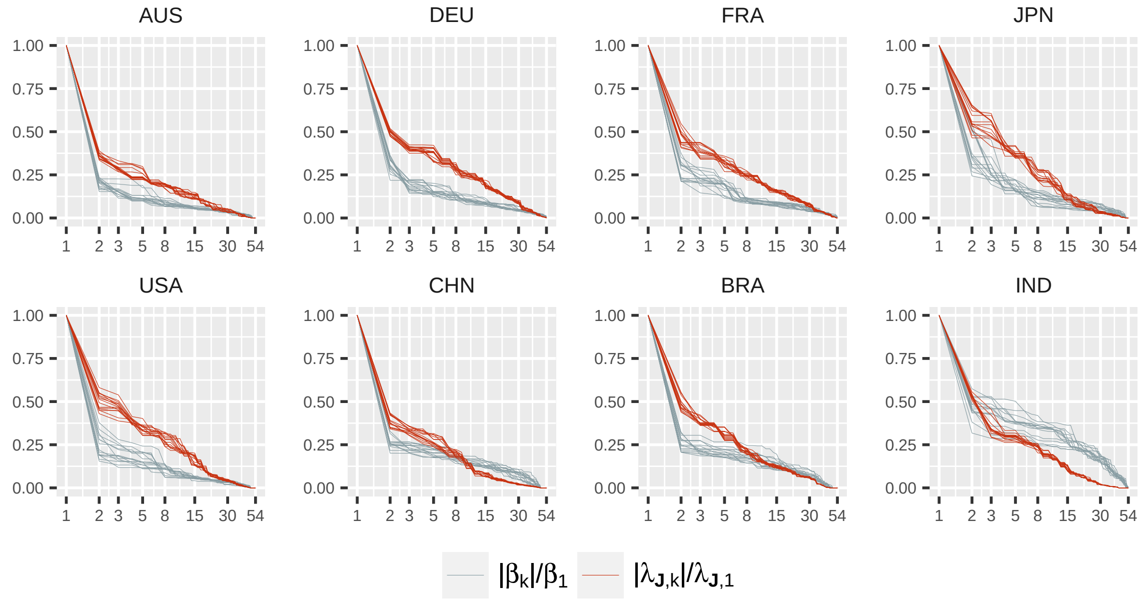


Figure 4: Rank plot of the moduli of the eigenvalues $|\lambda_{\mathbf{J}k}|/\lambda_{\mathbf{J}1}$ and the eigenlabours $|\beta_k|/\beta_1$ for 8 countries and 15 years (2000-2014). Y-axis scale is in log base 10. Source: authors' calculations based on the WIOD database, 2016 release.

shows the rank-plots for the 43 countries (figures C.10 and C.11) and the descriptive tables for years 2000 and 2014 (tables C.1 and C.2). With this evidence we shall show that for the vast majority of the economies the $|\beta_k|\beta_1$ display either faster rates of convergence towards zero than those of the $|\lambda_{\mathbf{J}k}|$ or at least similar rates for the first observations —the observations with the largest values.

The motivation for this exercise is twofold. Firstly, so far we have studied the statistical characteristics of the *subdominant* eigenlabours β_k and $|\beta_k|$ with no reference to the "dominant" $\beta_1 = |\beta_1|$. This is not the case for the eigenvalues where the $\lambda_{\mathbf{J}k\geq 2} = \lambda_{\mathbf{J}k\geq 2}/\lambda_{\mathbf{J}1}$. The $\beta_1 \in \mathbb{R}^+$ but, in contrast with $1 = \lambda_{\mathbf{J}1} \geq \lambda_{\mathbf{J}k\geq 2}$, the β_k are not bounded from above. In addition, the sequence $\beta_1, |\beta_2|, \ldots, |\beta_n|$ is not monotonically non-increasing, so in general we do not have $\beta_1 \geq |\beta_2| \geq \cdots \geq |\beta_n|$ as we have $\lambda_{\mathbf{J}1} > |\lambda_{\mathbf{J}2}| \geq \cdots \geq |\lambda_{\mathbf{J}n}|$. There might be cases where for some $k \geq 2$ we have $|\beta_k| > \beta_1$.[17] Therefore, it is relevant to study the $|\beta_{k\geq 2}|/\beta_1$. Secondly, the rank-plot has been widely used by the literature in the study of $|\lambda_{\mathbf{J}k}|$ and in advocating the existence of sufficiently small $\lambda_{\mathbf{J}k\geq 2}$. By studying the rank-plot of both the $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ we can evaluate the speed of convergence of $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ on the same footage.

Each line in Figure 4 represents the rank-plot for one sample, i.e., one country-year-variate. The Y-axis represents the value of the $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|/\beta_1$ and the X-axis represents the ordering from left to right of the variates defined as follows: For each sample, the first value is $1 = |\lambda_{\mathbf{J}1}|/\lambda_{\mathbf{J}1} = |\beta_1|/\beta_1$. The rest of the series

[17] For most of the economies, $\beta_1 > |\beta_{k\geq 2}|$. The grey rows in Table 1 show where this do not hold.

takes the remaining $n-1$ observations ($|\lambda_{\mathbf{J}k\geq 2}|$ or $|\beta_{k\geq 2}|/\beta_1$) and arranges them in decreasing order. The resulting lines are monotonically non-increasing. The X-axis scale is in logarithms, to zoom in on the first observations, the ones with the largest magnitudes.

Figure 4 shows that both $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|/\beta_1$ are subject to a fast rate of convergence with a remarkable time persistence. Starting from a value of 1, the second term drops abruptly and the remaining of the observations continue their reduction at fast rates such that they reach a narrow region of small values pretty fast. Plots in Appendix C.4 show the same patterns for the 43 countries.

When comparing the trends between the $|\lambda_{\mathbf{J}k}|$ and the $|\beta_k|/\beta_1$, for the vast majority of the 645 economies the rate of convergence of $|\beta_k|/\beta_1$ is at least as fast as that of $|\lambda_{\mathbf{J}k}|$. In Figure 4, the $|\beta_k|/\beta_1$ are bellow the $|\lambda_{\mathbf{J}k}|$ for most of the sample for all countries except for CHN, BRA and IND. For CHN and BRA the $|\beta_k|$ outperform the $|\lambda_{\mathbf{J}k}|$ for the observations with the largest values (2-6/8) and then the order inverts. For IND the $|\lambda_{\mathbf{J}k}|$ present a faster rate of decline. The fifth and sixth columns for $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ in Table 1 shows the 2nd and 3rd positions of the rank-plots for the 43 countries in 2011. They show that for the vast majority of the countries the rate of convergence of $|\beta_k|/\beta_1$ is considerably larger than that of $|\lambda_{\mathbf{J}k}|$: on average, the 2nd and 3rd positions of the ordering of the $|\lambda_{\mathbf{J}k}|$ are 0.53 and 0.4 *vis-a-vis* 0.37 and 0.26 for $|\beta_k|/\beta_1$.[18] These averages represent a decrease of 47% from the 1st to the 2nd position and 25% from the 2nd to the 3rd position in $|\lambda_{\mathbf{J}k}|$ *vis-a-vis* 63% and 31% in $|\beta_k|$.

When looking at the rank-plots of the 43 countries in Appendix C.4 we can distinguish four groups of countries. In the first one, composed of 24, all the rank-plots of the eigenlabours are bellow the eigenvalues for the first 15 to 30 positions (AUS, BEL, BGR, CAN, CZE, DEU, DNK, ESP, EST, FIN, FRA, GBR, ITA, JPN, LIT, NLD, NOR, POL, PRT, ROU, SVK, SVN, SWE, and USA). In the second group, composed of 5 countries, almost all years the eigenlabours are bellow the eigenvalues for the first observations (up to position 8) and from there on either they are indistinguishable or they flip (BRA, CHN, HRV, RUS, and TUR). In the third group, composed of 8 countries, the faster rate of convergence alternates between $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ according to the years of the sample (AUT, CHE, IDN, KOR, LVA) or they are indistinguishable (GRC, HUN, MEX). In the last group, composed of 6 countries, the $|\lambda_{\mathbf{J}k}|$ are bellow the $|\beta_k|$ either for all the years (LUX and TWN) or at least for one year (CYP, IND, IRL, and MLT).

[18] For year 2000 (2014) the second and third position of the ordering of the $|\lambda_{\mathbf{J}k}|$ is 0.52 and 0.41 (0.51 and 0.39) *vis-a-vis* 0.29 and 0.23 (0.37 and 0.26) for $|\beta_k|/\beta_1$ (see Tables C.1 and C.1).

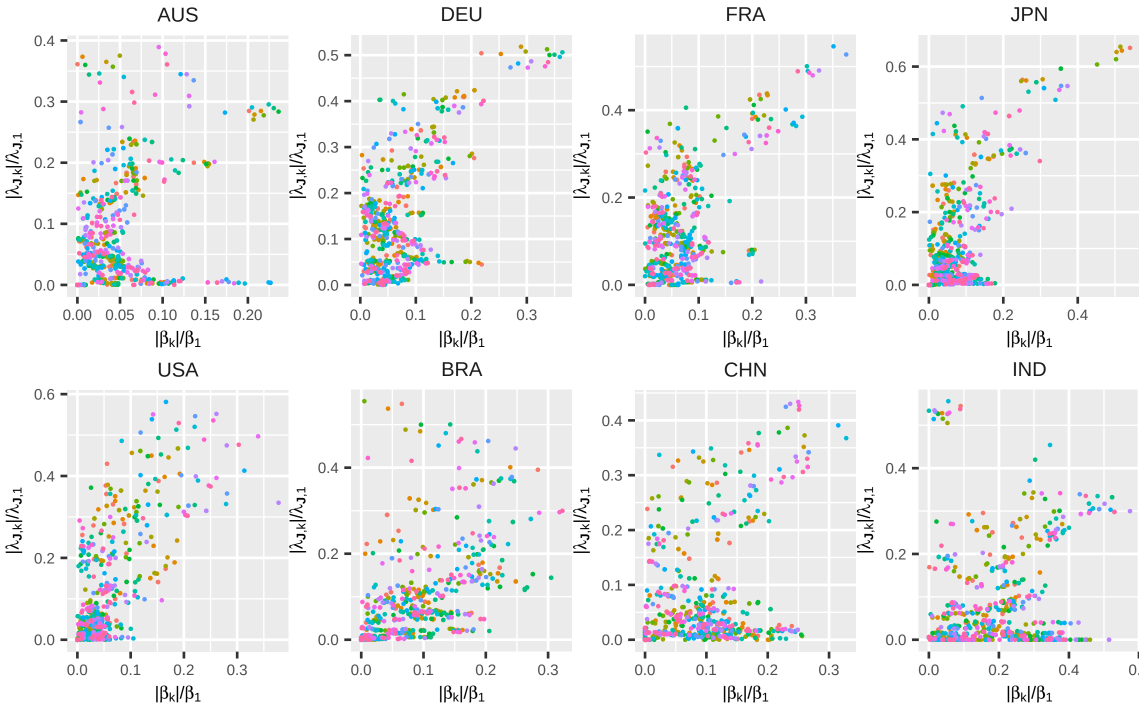


Figure 5: Scatter plot of the normalised moduli of the subdominant ($k \geq 2$) eigenvalues $|\lambda_{\mathbf{J}k}|/\lambda_{\mathbf{J}1}$ and $|\beta_k|/\beta_1$ for 8 countries and 15 years (2000-2014). Source: authors' calculations based on the WIOD database, 2016 release.

Complementarity. The scatter plot of $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|/\beta_1$ in Figure 5 allows us to study some aspects of the association between these two variates.[19] The plot shows that there exits a certain degree of complementarity between $\lambda_{\mathbf{J}k}$ and β_k. For instance, in AUS and IND, their largest $|\lambda_{\mathbf{J}k}|$ are associated with small $|\beta_{k\geq 2}|/\beta_1|$. The opposite situation also holds. Many of the largest $|\beta_k|/\beta_1$ for BRA and CHN are associated with low magnitudes of $|\lambda_{\mathbf{J}k}|$. Hence, there is some compensating process in the interaction between $\lambda_{\mathbf{J}k}$ and β_k in which larger values of one are reduced by the other in the products $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$.

3.3 Robustness

The novelty of the theoretical relevance of and the statistical regularities in the eigenlabours $\beta_k \equiv \mathbf{v}\mathbf{x}_k^T = \frac{\alpha_k}{1-\lambda_{\mathbf{A}k}}$ compels us to evaluate the robustness of the empirical results. The paper considers alternatives in the representations of the relationship between labour vectors and the input matrices as well as in the construction of the labour vector. However, space limitations forces us only to present a summary of the conclusions and refer the reader to Appendix D for detailed results.

First, Appendix D.1 shows that subdominant eigenlabours α_k and eigenvalues $\lambda_{\mathbf{A}k}$ also present persistent clustering around zero which increases as we approach

[19] We cannot associate the pairs $(|\lambda_{\mathbf{J}k}|, |\beta_k|/\beta_1)$ in Figure 4 because in general we do not have $\beta_1 \geq |\beta_2| \geq \cdots \geq |\beta_n|$.

zero from any direction. Here we also find that the $\lambda_{\mathbf{A}k}$ present less of a stronger tendency compared with the α_k. The main difference between the direct "eigens" $(\lambda_{\mathbf{A}k}, \alpha_k)$ *vis-a-vis* the vertically integrated "eigens" $(\lambda_{\mathbf{J}k}, \beta_k)$ is that there is higher variability in the former and higher magnitude in their maxima observations.

Second, Appendices D.2 constructs the labour vector without correcting for bias in the skills of labour. In this case the parameter of interest is β_k again, but the labour vector $\mathbf{l}$ is constructed taking persons engaged in production (employees + self-employed) instead of the skilled-adjusted labour input. The summary results in Appendix D.2 and the detailed results in Appendix D.3 supports the same conclusions derived for the skill-adjusted labour vector presented in sections 3 and 4: the robust evidence on the existence of stylised facts *SF1* and *SF3-SF6*.

4 Behind the regularities in the eigenlabours

Section 2.2 showed that the eigenlabours $\beta_k \equiv \mathbf{v}\mathbf{x}_k^T = \frac{\alpha_k}{1-\lambda_{\mathbf{A}k}}$ and $\alpha_k \equiv \mathbf{l}\mathbf{x}_k^T$ play a crucial role in deriving the conditions to obtain low-order polynomial approximations of price and capital value curves ($\lambda_{\mathbf{J}k}\beta_k \approx 0$ and $\lambda_{\mathbf{J}k}^2\beta_k \approx 0$, for $k \geq 2$). In addition, Sections 3.2 and 3.3 showed that there is a statistical tendency towards zero of the subdominant $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$ (*SF1*) and that the eigenlabours α_k and β_k play a crucial role due to their statistical tendency towards zero (*SF3*). But what constraints on the productive structure are behind *SF3*? From Equations (11)-(12) there are reasons to expect that if the labour vector $\mathbf{l}$ and the P-F eigenvector $\mathbf{y}_1$ were to be closely proportional, then we could expect small $\alpha_{k\geq 2}$ and $\beta_{k\geq 2}$. With this theoretical background, we now try to explain the regularities in the eigenlabours by assessing the degree of proportionality between $\mathbf{l}$ and $\mathbf{y}_1$. We can express the deviations from proportionality as:

$$[\xi_j^{\mathbf{l}}] \equiv \boldsymbol{\xi}^{\mathbf{l}} \equiv \lambda_{\mathbf{A}1}\mathbf{l} - \mathbf{l}\mathbf{A} = \sum\nolimits_{k=2}^{n}(\lambda_{\mathbf{A}1} - \lambda_{\mathbf{A},k})\alpha_k\mathbf{y}_k \tag{15}$$

$$[\xi_j^{\mathbf{v}}] \equiv \boldsymbol{\xi}^{\mathbf{v}} \equiv \mathbf{v} - \mathbf{v}\mathbf{J} = \sum\nolimits_{k=2}^{n}(1 - \lambda_{\mathbf{J}k})\beta_k\mathbf{y}_k \tag{16}$$

$$[\eta_j^{\mathbf{l}}] \equiv \boldsymbol{\eta}^{\mathbf{l}} \equiv \mathbf{l} - \alpha_1\mathbf{y}_1 = \sum\nolimits_{k=2}^{n}\alpha_k\mathbf{y}_k \tag{17}$$

$$[\eta_j^{\mathbf{v}}] \equiv \boldsymbol{\eta}^{\mathbf{v}} \equiv \mathbf{v} - \beta_1\mathbf{y}_1 = \sum\nolimits_{k=2}^{n}\beta_k\mathbf{y}_k = \sum\nolimits_{k=2}^{n}\frac{\alpha_k}{1-\lambda_{\mathbf{A}k}}\mathbf{y}_k. \tag{18}$$

In each case, $\mathbf{l}, \mathbf{v} \propto \mathbf{y}_1 \iff \alpha_{k\geq 2} = \beta_{k\geq 2} = 0 \iff \boldsymbol{\xi}^{\mathbf{l}} = \boldsymbol{\xi}^{\mathbf{v}} = \boldsymbol{\eta}^{\mathbf{l}} = \boldsymbol{\eta}^{\mathbf{v}} = \mathbf{0}$.

In order to support this conjecture, we study the empirical densities of the *coefficients* in Equations (15)-(18). Figure 6 presents the densities for the 8 countries of the MT2011 sample whereas figures C.12-C.15 in Appendix C contain those for the 43 countries. The red lines correspond to the densities of the deviation coefficients $\xi_j^{\mathbf{l}}$, $\xi_j^{\mathbf{v}}$, $\eta_j^{\mathbf{l}}$, and $\eta_j^{\mathbf{v}}$. The densities with the scaled labour coefficients $\lambda_{\mathbf{A}1}l_j$ and v_j are drawn with black lines. The grey densities correspond to the quantities of labour

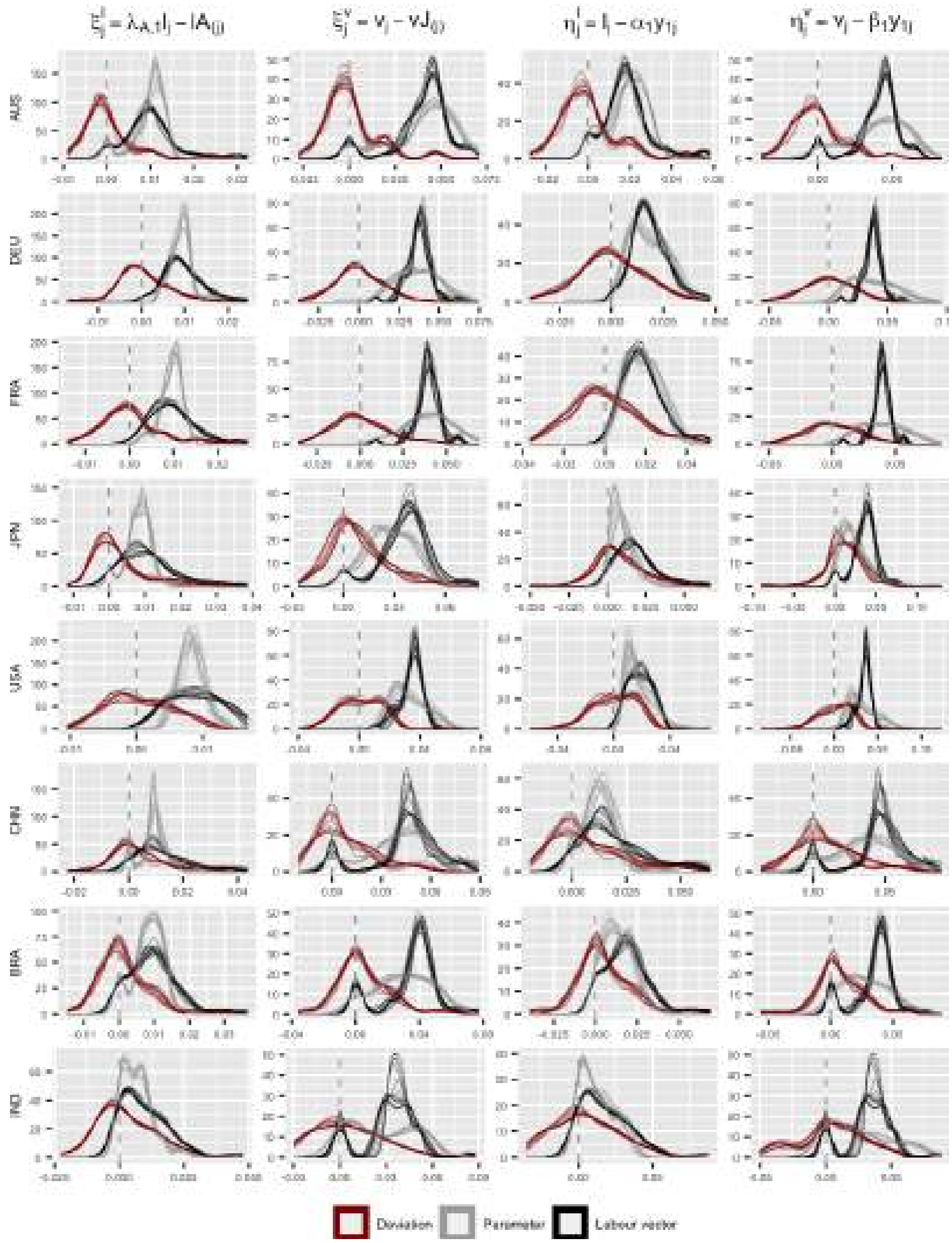


Figure 6: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi^{\mathbf{l}}_j, \xi^{\mathbf{v}}_j, \eta^{\mathbf{l}}_j, \eta^{\mathbf{v}}_j)$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

contained in the direct $\mathbf{lA}_{(j)}$ and the vertically integrated $\mathbf{vJ}_{(j)}$ means of production and the coefficients of the scaled P-F eigenvector $\alpha_1 y_{1j}$ and $\beta_1 y_{1j}$. There are 15 lines for each colour —one for each country-year-variate.

These figures provide robust evidence on the existence of the following stylised facts:

Country	$\xi^{\mathbf{l}}_j = \lambda_{\mathbf{A}1} l_j - \mathbf{lA}_{(j)}$ μ	$Q2$	σ	$\xi^{\mathbf{v}}_j = v_j - \mathbf{vJ}_{(j)}$ μ	$Q2$	σ	$\eta^{\mathbf{l}}_j = l_j - \alpha_1 y_{1j}$ μ	$Q2$	σ	$\eta^{\mathbf{v}}_j = v_j - \beta_1 y_{1j}$ μ	$Q2$	σ
AUS	0.0003	-0.0009	0.0062	0.0000	-0.0010	0.0141	-0.0001	-0.0014	0.0145	-0.0008	-0.0010	0.0171
DEU	0.0004	-0.0006	0.0063	0.0004	0.0008	0.0149	0.0004	0.0002	0.0159	0.0002	0.0017	0.0207
FRA	0.0008	-0.0001	0.0070	0.0015	-0.0002	0.0160	0.0018	0.0002	0.0163	0.0023	0.0006	0.0203
JPN	0.0032	0.0002	0.0084	0.0105	0.0071	0.0166	0.0103	0.0062	0.0163	0.0151	0.0132	0.0205
USA	0.0008	0.0005	0.0048	0.0022	0.0026	0.0135	0.0025	0.0038	0.0157	0.0030	0.0061	0.0205
CHN	0.0037	0.0019	0.0097	0.0080	0.0050	0.0175	0.0075	0.0012	0.0164	0.0101	0.0075	0.0216
BRA	0.0012	0.0000	0.0071	0.0045	0.0008	0.0161	0.0051	0.0013	0.0165	0.0070	0.0038	0.0191
IND	0.0033	0.0000	0.0123	0.0061	0.0000	0.0240	0.0059	0.0014	0.0244	0.0057	0.0049	0.0278
Average	0.0017	0.0000	0.0081	0.0042	0.0008	0.0171	0.0042	0.0012	0.0175	0.0053	0.0039	0.0216

Table 2: Summary statistics of the coefficients of the deviations vectors $\boldsymbol{\xi}^{\mathbf{l}}$, $\boldsymbol{\xi}^{\mathbf{v}}$, $\boldsymbol{\eta}^{\mathbf{l}}$, and $\boldsymbol{\eta}^{\mathbf{v}}$ for 8 countries, 2011. Mean (μ), median ($Q2$), and standard deviation (σ). Source: authors' calculations based on the WIOD database, 2016 release.

SF4 the densities are smooth and unimodal with high degree of symmetry in the IQR.

SF5 the empirical densities are time invariant.

SF6 the central value of the densities of the deviations are located in a small vecinity around zero.

It is a remarkable result that the coefficients of each of the three vectors appearing in Equations (15)-(18) display reduced variability and tend to concentrate around a central value. As we approach this central value, which corresponds to the mode of the distribution, there is an increasing concentration of the observations.[20] Although there is heterogeneity in kurtosis, in location, and in symmetry, most of the densities conform to some well behaved distribution. Whatever that distribution is, the shape hardly changes across time, implying the temporal reproducibility of the constraints in the productive structure. These features characterise stylised facts *SF4* and *SF5*.

Figure 6 shows that the deviation coefficients $(\xi^{\mathbf{l}}_j, \xi^{\mathbf{v}}_j, \eta^{\mathbf{l}}_j, \eta^{\mathbf{v}}_j)$ are persistently centred around zero (*SF6*). This conclusion is corroborated by Table 2 which shows for the MT2011 sample that their mean and median are extremely low and have limited variability. One might wonder if the location of the distribution near zero is the outcome of the small coefficients of the two vectors used to construct the deviations. This is nonetheless ill-founded. First, the $(l_j, v_j, \mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$ are all nonnegative which is not true for the deviation coefficients. Second, Table 3 shows that although their median is small, it is considerably higher than the median of the deviation coefficients. For instance, the average median for $\xi^{\mathbf{l}}_j$ is 0.0000 and that for $\lambda_{\mathbf{A}1} l_j$ and $\mathbf{lA}_{(j)}$ are 0.0090 and 0.0093 —the black and grey densities are located considerably to the right of zero. Third, and most importantly, *SF6* emerges irrespective of the similarity of each pair of black and grey densities. For instance, in FRA-$\eta^{\mathbf{l}}_j$ the densities of l_j and $\alpha_1 y_{1,j}$ are remarkably similar and their deviations have a density which is less peaked, more variable, and not-as-close to zero as in the

[20] Bimodality in the black and grey densities arises due to some industries with non information.

case of FRA-$\xi^{\mathbf{l}}_j$ even though $\lambda_{\mathbf{A}1} l_j$ and $\mathbf{lA}_{(j)}$ have less similar distributions.[21]

	$\boldsymbol{\xi}^{\mathbf{l}} = \lambda_{\mathbf{A},1}\mathbf{l} - \mathbf{lA}$		$\boldsymbol{\xi}^{\mathbf{v}} = \mathbf{v} - \mathbf{vJ}$		$\boldsymbol{\eta}^{\mathbf{l}} = \mathbf{l} - \alpha_1 \mathbf{y}_1$		$\boldsymbol{\eta}^{\mathbf{v}} = \mathbf{v} - \beta_1 \mathbf{y}_1$	
Country	$\lambda_{\mathbf{A},1} l_j$	$\mathbf{lA}_{(j)}$	v_j	$\mathbf{vJ}_{(j)}$	$\alpha_1 y_{1j}$	l_j	$\beta_1 y_{1j}$	v_j
AUS	0.0106	0.0096	0.0418	0.0440	0.0198	0.0180	0.0423	0.0440
DEU	0.0097	0.0089	0.0372	0.0390	0.0160	0.0166	0.0344	0.0390
FRA	0.0096	0.0093	0.0400	0.0406	0.0167	0.0167	0.0378	0.0406
JPN	0.0085	0.0104	0.0258	0.0389	0.0065	0.0168	0.0169	0.0389
USA	0.0079	0.0088	0.0288	0.0357	0.0114	0.0183	0.0220	0.0357
CHN	0.0093	0.0101	0.0407	0.0482	0.0113	0.0147	0.0357	0.0482
BRA	0.0091	0.0092	0.0329	0.0390	0.0133	0.0181	0.0270	0.0390
IND	0.0066	0.0078	0.0249	0.0326	0.0108	0.0147	0.0229	0.0326
Average	0.0090	0.0093	0.0349	0.0392	0.0133	0.0170	0.0306	0.0392

Table 3: Median of the vectors defining the deviation vectors for 8 countries, 2011. Source: authors' calculations based on the WIOD database, 2016 release.

Stylised facts *SF4*-*SF6* in coefficients $(\xi^{\mathbf{l}}_j, \xi^{\mathbf{v}}_j, \eta^{\mathbf{l}}_j, \eta^{\mathbf{v}}_j)$ shows the operation of cancelling effects between the deviations and point to the statistical tendency of average deviations towards zero. Figures D.16-D.19 in Appendix D.2 show that *SF4*-*SF6* are robust when using persons engaged as the labour unit.

5 Implications

Summary of empirical results. Section 3 showed that for 43 countries in 2000-14 *there is robust empirical evidence of a statistical tendency of subdominant* $\lambda_{\mathbf{J}k}\beta_k$ *and* $\lambda^2_{\mathbf{J}k}\beta_k$ *towards zero* (the stylised fact (*SF*) 1, *SF1*). In each economy there is a clustering of the *subdominant* $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ in a small neighbourhood around zero, with an intensifying clustering as we approach the origin from any direction. Observations outside this narrow region can be considered outliers and only a few of them exist. For the vast majority of the economies, the magnitude of these extreme and scarce observations is rather small. The subdominant $\lambda_{\mathbf{J}k\geq 2}$ and $\beta_{k\geq 2}$ present the same general patterns as the $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, although less pronounced and with important differences between them. These small qualifications are nonetheless of paramount importance for the properties of price curves.

On the one hand, there are *many* $\lambda_{\mathbf{J}k\geq 2}$ with a magnitude *considerably greater than zero* (*SF2*). There is no evidence that the $\lambda_{\mathbf{J}k\geq 2}$ are the sole responsible for the tendency towards zero of the subdominant $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$. The $\beta_{k\geq 2}$, on the other hand, contributes significantly to achieve the latter tendency. The $\beta_{k\geq 2}$ display a stronger tendency towards zero and, for the vast majority of the economies, at a faster rate than that of the $\lambda_{\mathbf{J}k\geq 2}$ (*SF3*). Therefore, it is the joint action of the $\lambda_{\mathbf{J}k\geq 2}$ and $\beta_{k\geq 2}$ what is behind the *SF1*. *SF1* and *SF3* complement the regularities in the productive structures already identified in the literature.

[21] A subtraction like $l_j - \alpha_1 y_{1,j}$ does not necessarily correspond to l_j and $\alpha_1 y_{1,j}$ with similar values.

In Section 4 it was conjectured that the statistical patterns in the $\beta_{k\geq 2}$ can be the outcome of the statistical tendency towards the proportionality between the labour coefficient vectors $(\mathbf{l}, \mathbf{v})$ and the left-hand P-F eigenvector $\mathbf{y}_1$ of matrices $(\mathbf{A}, \mathbf{J})$. The deviations from proportionality cluster around zero and have unimodal empirical distributions with a limited degree of variability and some symmetry (*SF4*-*SF6*).

Identified constraints on the productive structures behind the regularities in empirical price curves. Section 2 showed that the subdominant eigenlabours $\beta_{k\geq 2}$ can play a role as important as that of the subdominant eigenvalues $\lambda_{\mathbf{J}k\geq 2}$ eliminating the sources of nonlinearities of price and capital curves. The statistical tendency towards $\beta_{k\geq 2} \approx 0$ for the 645 economies in the WIOD database (*SF3*) confirms their importance, so that the relationship between the labour vector and the input matrix is a feature of the productive structures that cannot be overlooked. Based on *SF1*-*SF3* we conjecture that *it is the joint action of the eigenvalues and the eigenlabours what is behind the monotonicity/near monotonicity in empirical price and capital intensity curves reported in the literature.*[22]

On some empirical evidence *against* the close proportionality between the vectors of labour coefficients $(\mathbf{l}, \mathbf{v})$ and $\mathbf{y}_1$, the P-F eigenvector of matrices $(\mathbf{A}, \mathbf{J})$. Some authors in the literature conclude that there are considerable deviations between $(\mathbf{l}, \mathbf{v})$ and $\mathbf{y}_1$.[23] This conclusion contrast with (1) the conjecture on the close relation of proportionality between these vectors in Torres-González (2020) and in this paper and (2) the hypothesis of zero average deviations advanced by Schefold (2013a, 2016) (see also the next topic).

The latter authors base their conclusion on the empirical evaluation of the deviations between $(\mathbf{l}, \mathbf{v})$ and $\mathbf{y}_1$ using different scalar indicators (e.g. the mean absolute deviation, MAD). In addition, Tsoulfidis (2021, p. 107) computes correlations and evaluates the scatter plot of the coefficients of $\mathbf{l}$ and $\mathbf{y}_1$ concluding that there is a "rather weak relationship between the two vectors". In contrast, this paper conjectures that the statistical properties of the subdominant eigenlabours α_k and β_k and the empirical densities of the deviations between $(\mathbf{l}, \mathbf{v})$ and $\mathbf{y}_1$ reflects a close relation of proportionality between $(\mathbf{l}, \mathbf{v})$ and $\mathbf{y}_1$. Therefore, future research must address this seemingly contradictory evidence and how it connects with the explanation of the empirical regularities in price curves.

[22] Given that constraints $\lambda_{\mathbf{J}k}\beta_k \approx 0$ ($\lambda^2_{\mathbf{J}k}\beta_k \approx 0$), for $k = 2, \ldots, n$, are sufficient conditions for approximately linear (quadratic) price and constant (linear) capital value curves, *SF1* provides robust evidence that prices and capital intensities curves in the economies of the WIOD database would display nearly linear/quadratic and constant/linear shapes, respectively.

[23] "[$\mathbf{y}_1$] deviates *considerably* from [$\mathbf{l}$]" (Iliadi et al., 2014, p. 47). See also Mariolis and Tsoulfidis (2016a, p. 306), Mariolis et al. (2021, p. 122), and Tsoulfidis (2021, p. 106-8).

Implications of the empirical results to other strands in the literature.

Evidence in favour of Schefold's third hypothesis to derive linear wage-profit curves. Schefold (2013a,b) proposes an explanation of a closely related regularity in empirical production-price models, namely, the near linearity in wage curves. One of the hypotheses he advances, his third hypothesis in deterministic systems (see Schefold, 2013a, sec. 4), consists on zero-average deviations between the labour vector $\mathbf{l}$ and $\mathbf{y}_1$. In (17) we defined $[\eta_j^l] \equiv \boldsymbol{\eta}^{\mathbf{l}} \equiv \mathbf{l} - \alpha_1 \mathbf{y}_1 = \sum_{k=2}^{n} \alpha_k \mathbf{y}_k$ and in Section 4 we studied some of its statistical properties. Let the average value of the $\eta_j^{\mathbf{l}}$ be $\bar{\eta}^{\mathbf{l}} \equiv \frac{1}{n} \sum_{j=1}^{n} \eta_j^{\mathbf{l}} = \frac{1}{n} \boldsymbol{\eta}^{\mathbf{l}} \mathbf{e}^T$.

Table 2 reports close to zero values of $\bar{\eta}^{\mathbf{l}}$: for the MT2011 sample, the $\bar{\eta}^{\mathbf{l}}$ are between $(-0.0001, 0.0103)$, with a mean value of 0.0042. The medians in Table 2 and the density plots in Figure 6 show that the deviations η_j^l are centred around zero, with a limited degree of variability and considerable symmetry. These small values of $\bar{\eta}^{\mathbf{l}}$ are not the outcome of the small scale of the vectors and matrices: On average, the median values of $\alpha_1 y_{1j}$ and l_j are 0.0133 and 0.0170, respectively, which are a little more that 11 and 14 times higher than the median of the η_j^l (see Table 3). The location of the density plots of $\alpha_1 y_{1j}$ and l_j in Figure 6 are considerably to the right relative to the density plots of η_j^l.

The density plots of the full WIOD database (see figures C.12-C.15 and D.16-D.19 in appendices C.5 and D.3) provide robust evidence towards Schefold's third hypothesis. Given that $\bar{\eta}^{\mathbf{l}} = \sum_{k=2}^{n} \alpha_k \bar{y}_k$, where $\bar{y}_k \equiv \frac{1}{n} \sum_{j=1}^{n} y_{k,j}$ we can conclude that the statistical tendency towards $\alpha_{k \geq 2} \approx 0$ reported in Section 3.3 contributes to the close-to-zero values in $\bar{\eta}^{\mathbf{l}}$. It is left for future research the role played by average values of the eigenvectors, $\bar{y}_k$.

Identification of constraints in productive structures behind the empirical closeness between production prices and direct prices. One stylised fact in the literature on empirical prices of production, to which currently there is no explanation generally accepted, is that scalar measures of the deviations between production prices $\mathbf{p}(\rho) \equiv [p_j(\rho)]$ computed at the observed relative profit rate of the economy and prices proportional to the quantities of embodied labour $\mathbf{v} \equiv [v_j]$, frequently called direct prices, are small.[24] One widely used measure is the mean absolute deviations (MAD). Let $\mathbf{p}(\rho) - \mathbf{v} \equiv \boldsymbol{\chi} \equiv [\chi_j]$ be the vector of deviations and $\bar{\chi} = \frac{1}{n} \sum_{j=1}^{n} \chi_j$ their average. Then,

$$MAD(\boldsymbol{\chi}) \equiv \frac{1}{n} \sum_{j=1}^{n} |\chi_j - \bar{\chi}| \,.$$

[24] Işıkara and Mokre (2021) provide evidence for the full WIOD database. See also Mariolis and Tsoulfidis (2016b, chap. 3) and Shaikh (2016, chap. 9). When the standard commodity $\mathbf{y}^s$ is the numéraire $\mathbf{p}(\rho)\mathbf{y}^s = 1$ and total labour equals $1 = \mathbf{v}\mathbf{y}^s$, then direct prices *equal* $\mathbf{v}$.

Appendix E presents the spectral representation of $MAD(\boldsymbol{\chi})$ and shows that it can be made dependent on four factors: the relative profit rate $\rho \equiv \frac{r}{R}$, the subdominant eigenvalues $\lambda_{\mathbf{J}k\geq 2}$, the subdominant eigenlabours $\beta_{k\geq 2}$, and the differences between the coefficients of eigenvector $y_{k,j}$ and its average value $\bar{y}_k$, for $k = 2, \ldots, n$. Specifically,

$$MAD(\boldsymbol{\chi}) \equiv \frac{1}{n}\sum\nolimits_{j=1}^{n}\left|\sum\nolimits_{k=2}^{n}\frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k\left[y_{k,j}-\bar{y}_k\right]\right| \tag{19}$$

$$\leq \sum\nolimits_{k=2}^{n}|\rho|\left|\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})}\right|\left|\frac{\beta_k}{\mathbf{y}_k\mathbf{x}_k^T}\right|\cdot MAD(\mathbf{y}_k) \tag{20}$$

Equations (19) and (20) show the existence of a link between the factors affecting the magnitude of $MAD(\boldsymbol{\chi})$ and the shapes in price curves: we can reduce the sources of nonlinearities in price curves (see (6)) and the magnitude of $\mathrm{MAD}(\boldsymbol{\chi})$ as much as we want by reducing $\beta_{k\geq 2}$. Now, Mariolis and Tsoulfidis argue that "for realistic values of the relative rate of profit [ρ] ... the traditional measures of production price-labour value deviations (i.e. the MAD, RMS%E and MAWD) ... and the 'd- distance' ... *tend to be close to each other*" (2010, p. 709; our emphasis). Hence, the statistical tendency towards $\beta_{k\geq 2} \approx 0$ found for the U.S. (see Torres-González, 2020) and the WIOD database is one potential constraint producing the reported small measures of deviations between production prices and direct prices. Torres-González (2020) argues that this pattern in $\beta_{k\geq 2}$ is related with the statistical tendency of capital intensities to cluster around central values with a limited variability irrespective of the profit rate. Hence, we might have the inputs to provide a satisfactory explanation for this stylised fact.

Additional sources dimensionality reduction of empirical multisector systems. It has been argued in the literature that multisectoral systems of production constructed with data from the input-output accounts from actual economies can be compressed into reduced-rank systems without loosing relevant information.[25] That is, beginning with the input-coefficient matrix and labour-coefficient vector constructed from the economic accounts, we can impose constraints on the productive structure consistent with monotonic/near monotonic price and capital curves and obtain low dimensional n-industries systems which are formally equivalent to traditional 2- or 3-industries theoretical models. In the corn-tractor model, for instance, one industry produces means of production ("tractors") whereas the second one only consumption goods ("corn"). In the low dimensional n-industries system, an equivalent transformation of the model can make it formally equivalent to this 2-industries model: one *composite* industry (the "tractor" industry) producing means of production and the

[25] E.g. Mariolis and Tsoulfidis (2016a, pp. 303, 314, 317) and Mariolis and Tsoulfidis (2018, pp. 5, 8, 11). Schefold's (2013a, sec. 5) deterministic or perturbed 'one-industry systems' is a closely related idea.

other $n-1$ industries producing consumption commodities.

The empirical results from the WIOD database imply that the statistical structure allowing the efficient information compression does not come only from the strong proportionality between the columns of matrix $\mathbf{J}$, but also from the sufficiently strong proportionality between $\mathbf{l}$ and $\mathbf{y}_1$. The empirical evidence for low-dimension models is more robust now.

6 Conclusions

The analysis of the production-price model showed that the behaviour of industries' prices and capital value as an effect of hypothetical changes in distribution, i.e., price an capital curves, depends not only on the characteristics of the eigenvalues of the input matrix, but also on the relationship between the labour vector and this matrix. Based on these theoretical results, the paper studied 43 economies for 15 years from the WIOD database and showed that in every case the conditions for price and capital value curves to be well approximated by linear or quadratic functions are given by the statistical tendency towards the proportionality (1) between the columns of the input matrices and (2) between the labour vectors and the Perron-Frobenius eigenvectors of the input matrices. There is no evidence that the statistical behaviour of the eigenvalues, by themselves, can produce the monotonic/near monotonic curves that are persistently observed. The empirical regularities in these curves must be driven by the empirical regularities in the proportions between means of production and labour of all industries —not only in the proportions of the means of production. Hence, the paper generalises the characterisation of the curves from previous work in the literature and complements their set of constraints responsible for the empirical regularities.

While advances have been made in the *identification* of constraints in the productive structure behind the empirical regularities in price models and in *alternative representations* of these constraints (in terms of vertical integration, normalised general coordinates, and control systems), the *economic* explanation for the stylised facts has largely been disregarded. Why are the columns of the input matrices highly proportional? What market forces, innovation restrictions, and behavioural features of agents constrain the productive structure in such a way that the labour vector and the P-F eigenvector of the matrix are highly proportional? It is clear that technical change has been constrained in such a way as to produce the set of stylised facts reported in the paper and elsewhere. But why and how? These stylised facts must be considered in the construction of this economic explanation.

Acknowledgments. Preliminary results were presented at the School of Advanced Studies in the Reappraisal of the Surplus Approach, Roma Tre University, July 2018;

in the EEA Conference, 2018; the Seminario de Análisis Insumo-Producto, UNAM, January 2021; and the Grupo de Investigación en Dinámica Económica (GIDE) of the Universidad de la República, September 2021. The paper benefited from comments of the audience. Mathematical results in Appendix A benefited from the discussion with Adan Salas.

Funding. Jacobo Ferrer acknowledges financial support from the Fulbright Program-Spain, the Harry Magdoff and Paul Sweezy Fellowship, and The New School for Social Research. Daniel Torres acknowledges financial support from the Dirección General de Asuntos del Personal Académico (DGAPA) of UNAM (postdoc scholarship).

Conflicts of Interest. To the best of our knowledge, we have no conflicts of interest to disclose.

References

Bidard, C. (2004). *Prices, Reproduction, Scarcity.* Cambridge, UK, Cambridge University Press.

Bienenfeld, M. (1988). Regularity in price changes as an effect of changes in distribution. *Cambridge Journal of Economics*, 12(2):247–255.

Chilcote, E. B. (1997). *Interindustry Structure, Relative Prices, and Productivity: An Input-Output Study of the U.S. and O.E.C.D. Countries.* PhD thesis, The New School for Social Research.

Dietzenbacher, E., Los, B., Stehrer, R., Timmer, M., and de Vries, G. (2013). The construction of world input-output tables in the wiod project. *Economic Systems Research*, 25(1):71–98.

Gurgul, H. and Wojtowicz, T. (2015). On the economic interpretation of the bródy conjecture. *Economic Systems Research*, 27(1):122–131.

Han, Z. and Schefold, B. (2006). An empirical investigation of paradoxes: Reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics*, 30(5):737–765.

Iliadi, F., Mariolis, T., Soklis, G., and Tsoulfidis, L. (2014). Bienenfeld's approximation of production prices and eigenvalue distribution: Further evidence from five european economies. *Contributions to Political Economy*, 33(1):35–54.

Işıkara, G. and Mokre, P. (2021). Price-value deviations and the labour theory of value: Evidence from 42 countries, 2000–2017. *Review of Political Economy*, 34(1):165–180.

Kersting, G. and Schefold, B. (2021). Best techniques leave little room for substitution. a new critique of the production function. *Structural Change and Economic Dynamics*, 58:509–533.

Mariolis, T. (2021). Competing theories of value: A spectral analysis. *Contributions to Political Economy*, 40(1):1–25.

Mariolis, T., Rodousakis, N., and Soklis, G. (2021). *Spectral Theory of Value and Actual Economies. Controllability, Effective Demand, and Cycles*. Tokyo, Japan. Springer.

Mariolis, T. and Tsoulfidis, L. (2009). Decomposing the Changes in Production Prices into "Capital-Intensity" and "Price" Effects: Theory and Evidence from the Chinese Economy. *Contributions to Political Economy*, 28(1):1–22.

Mariolis, T. and Tsoulfidis, L. (2010). Measures of production price-labour value deviation and income distribution in actual economies: A note. *Metroeconomica*, 61(4):701–710.

Mariolis, T. and Tsoulfidis, L. (2011). Eigenvalue distribution and the production price-profit rate relationship in linear single-product systems: Theory and empirical evidence. *Evolutionary and Institutional Economics Review*, 8(1):87–122.

Mariolis, T. and Tsoulfidis, L. (2014). On bródy's conjecture: Theory, facts and figures about instability of the us economy. *Economic Systems Research*, 26(2):209–223.

Mariolis, T. and Tsoulfidis, L. (2016a). Capital theory 'paradoxes' and paradoxical results. resolved or continued? *Evolutionary and Institutional Economics Review*, 13(2):297–322.

Mariolis, T. and Tsoulfidis, L. (2016b). *Modern Classical Economics and Reality. A Spectral Analysis of the Theory of Value and Distribution*. Tokyo,Japan. Springer.

Mariolis, T. and Tsoulfidis, L. (2018). Less is more. capital theory and almost irregular-uncontrollable actual economies. *Contributions to Political Economy*, 37(1):65–88.

Ochoa, E. M. (1984). *Labor Values and Prices of Production. An Interindustry Study of the US Economy, 1947-1972*. PhD thesis, The New School for Social Research.

Pasinetti, L. L. (1977). *Lectures on the Theory of Production*. New York, NY, Columbia University Press.

Petri, F. (2021). What remains of the cambridge critique? on professor schefold's theses. Centro Sraffa Working Papers n. 50, September, ISSN: 2284-2845.

Scharnhorst, K. (2001). Angles in complex vector spaces. *Acta Applicandae Mathematicae*, (69):95–103.

Schefold, B. (1976). Relative prices as a function of the rate of profit: A mathematical note. *Zeitschrift fur Nationolokonomie*, 36(1-2):21–48.

Schefold, B. (2013a). Approximate surrogate production functions. *Cambridge Journal of Economics*, 37(5):1161–1184.

Schefold, B. (2013b). Only a few techniques matter! on the number of curves on the wage frontier. In Levrero, E. S., Palumbo, A., and Stirati, A., editors, *Sraffa and the Reconstruction of Economic Theory*, volume I. Theories of Value and Distribution, chapter 3. New York, NY, Palgrave Macmillan.

Schefold, B. (2016). Profits equal surplus value on average and the significance of this result for the marxian theory of accumulation. *Cambridge Journal of Economics*, 40(1):165–199.

Shaikh, A. (1998). The empirical strength of the labor theory of value. In Bellofiore, R., editor, *Marxian Economics. A Reappraisal*, volume II Profit, Prices and Dynamics, chapter 15. New York, NY. Macmillan St. Martins Press.

Shaikh, A. (2012). The empirical linearity of sraffa's critical output-capital rations. In Gehrke, C., Salvadori, N., Steedman, I., and Sturn, R., editors, *Classical Political Economy and Modern Theory. Essays in honour of Heinz Kurz*, chapter 6. New York, NY, Routledge.

Shaikh, A. (2016). *Capitalism: Competition, Conflict, Crises.* New York, NY, Oxford University Press.

Shaikh, A. (2020). An empirically sufficient form for sraffa prices. mimeo.

Shaikh, A., Pires, L. N., and Coronado, J. A. (2020). Eigenvalue distribution, matrix size and the linearity of wage-profit curves. Working Paper 14/2020, Department of Economics, The New School for Social Research.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities.* London, UK. The Syndics of the Cambridge University Press.

Timmer, M. P., Dietzenbacher, E., Los, B., Stehrer, R., and de Vries, G. (2015). An illustrated user guide to the world input-output database: the case of global automotive production. *Review of International Economics*, 23:575–605.

Timmer, M. P., Los, B., Stehrer, R., and de Vries, G. J. (2016). An anatomy of the global trade slowdown based on the wiod 2016 release. GGDC research memorandum number 162, University of Groningen.

Torres-González, L. D. (2017). Regularities in prices of production and the concentration of compositions of capitals. Working Paper 09/2017, Department of Economics, The New School for Social Research.

Torres-González, L. D. (2020). The characteristics of the productive structure behind the empirical regularities in prices of production. Working Paper 16/2020, Department of Economics, The New School for Social Research.

Torres-González, L. D. and Yang, J. (2019). The persistent statistical structure of the us input-output coefficient matrices: 1963-2007. *Economic Systems Research*, 31(4):481–504.

Tsoulfidis, L. (2021). *Capital Theory and Political Economy. Prices, Income Distribution and Stability.* Routledge.

Tsoulfidis, L. (2022). Capital theory debates: New developments and direction. *Investigación Económica*, 81(319):3–36.

Zambelli, S. (2018). The aggregate production function is NOT neoclassical. *Cambridge Journal of Economics*, 42(2):383–426.

A Polynomial approximations to price and capital value curves

The following results are developed from Bienenfeld (1988). A preliminary version of them were introduced in Torres-González (2020).

Let us re-express price curves (4) as

$$\mathbf{p}(\rho) = \mathbf{v} + \rho\mathbf{f} + \cdots + \rho^{q_0}\mathbf{f}^{q_0} + \mathbf{d}_{q_0}, \tag{A.1}$$

where $\mathbf{d}_{q_0} \equiv \sum_{s=q_0+1}^{+\infty} \rho^s \mathbf{f}^s$ is a series and $q_0 \in \{1, 2, \dots\}$. The exact or approximated order q_0 of the price polynomial depends on the characteristics of $\mathbf{d}_{q_0}$, in particular if $\mathbf{d}_{q_0} = \mathbf{0}$ or $\mathbf{d}_{q_0} \approx \mathbf{0}$.

The first thing to notice is that the succession $\rho^q \mathbf{f}^q$ is convergent, i.e., $\lim_{q\to+\infty} \rho^q \mathbf{f}^q = \mathbf{0}$. This convergence is secured by two factors. First, given that $0 \leq \rho = \frac{r}{R} < 1$, then $\lim_{q\to+\infty} \rho^q = 0$. Second, because $1 = \lambda_{\mathbf{J}1} > |\lambda_{\mathbf{J}k}| \geq 0$, for $k \geq 2$, this implies that $\mathbf{0} = \mathbf{y}_1\mathbf{J} - \mathbf{y}_1 = \mathbf{y}_1(\mathbf{J} - \mathbf{I})$ and because $\lim_{q\to+\infty} \mathbf{J}^q = \lim_{q\to+\infty} \sum_{k=1}^n \frac{\lambda_{\mathbf{J},k}^q}{\mathbf{y}_k\mathbf{x}_k^T}\mathbf{x}_k^T\mathbf{y}_k = \lim_{q\to+\infty} \frac{\lambda_{\mathbf{J}1}^q}{\mathbf{y}_1\mathbf{x}_1^T}\mathbf{x}_1^T\mathbf{y}_1 + \lim_{q\to+\infty} \sum_{k=2}^n \frac{\lambda_{\mathbf{J}k}^q}{\mathbf{y}_k\mathbf{x}_k^T}\mathbf{x}_k^T\mathbf{y}_k = \frac{1}{\mathbf{y}_1\mathbf{x}_1^T}\mathbf{x}_1^T\mathbf{y}_1$, so

$$\lim_{q\to+\infty} \mathbf{f}^q = \lim_{q\to+\infty} \left[\mathbf{v}\mathbf{J}^{q-1}(\mathbf{J} - \mathbf{I})\right] = \frac{\mathbf{v}\mathbf{x}_1^T}{\mathbf{y}_1\mathbf{x}_1^T}\mathbf{y}_1(\mathbf{J} - \mathbf{I}) = \mathbf{0}.$$

Second, $\mathbf{f}^q \geq \mathbf{f}^{q+1}$. This is, because $\lambda_{\mathbf{J}k}^1 \geq \lambda_{\mathbf{J}k}^2 \geq \cdots$ for all $k = 1, \dots, n$, then $(\lambda_{\mathbf{J}k} - 1)\lambda_{\mathbf{J}k}^q\beta_k\mathbf{y}_k \geq (\lambda_{\mathbf{J}k} - 1)\lambda_{\mathbf{J}k}^{q+1}\beta_k\mathbf{y}_k$. This implies that if $\mathbf{f}^q \approx \mathbf{0}$, then $\mathbf{f}^{q+1} \approx \mathbf{0}$.

In spite of $\lim_{q\to+\infty} \rho^q\mathbf{f}^q = \mathbf{0}$ and $\mathbf{f}^q \approx \mathbf{0}$, the series $\mathbf{d}_{q_0} \equiv \sum_{s=q_0+1}^{+\infty} \rho^s\mathbf{f}^s$ might not be convergent or the magnitude to which it converges might be considerable. In order to study the conditions under which $\mathbf{d}_{q_0} = \mathbf{0}$ or its norm $\|\mathbf{d}_{q_0}\|$ is sufficiently small, let us express $\mathbf{d}_{q_0}$ as

$$\begin{aligned}\mathbf{d}_{q_0} \equiv \sum_{s=q_0+1}^{+\infty} \rho^s\mathbf{f}^s &= \sum_{s=q_0+1}^{+\infty} \left[\rho^s \sum_{k=2}^n \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k\mathbf{x}_k^T}\lambda_{\mathbf{J}k}^{s-1}\beta_k\mathbf{y}_k\right] \\ &= \rho\sum_{k=2}^n \left(\sum_{s=q_0}^{+\infty} (\rho\lambda_{\mathbf{J}k})^s\right) \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k\mathbf{x}_k^T}\beta_k\mathbf{y}_k.\end{aligned}$$

But $|\rho|, |\lambda_{\mathbf{J}k}| < 1$, so $|\rho\lambda_{\mathbf{J}k}| < 1$ and the series in parenthesis converges to

$$\sum_{s=q_0}^{\infty} (\rho\lambda_{\mathbf{J}k})^s = (\rho\lambda_{\mathbf{J}k})^{q_0} \sum_{s=0}^{\infty} (\rho\lambda_{\mathbf{J}k})^s = \frac{\rho^{q_0}\lambda_{\mathbf{J}k}^{q_0}}{1 - \rho\lambda_{\mathbf{J}k}}.$$

Hence,

$$\begin{aligned}\mathbf{d}_{q_0} &= \rho^{q_0+1}\sum_{k=2}^n \frac{\lambda_{\mathbf{J}k} - 1}{(1 - \rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\lambda_{\mathbf{J}k}^{q_0}\beta_k\mathbf{y}_k \\ &= \rho^{q_0+1}\sum_{k=2}^n \gamma_k\mathbf{y}_k \\ \gamma_k &\equiv \frac{\lambda_{\mathbf{J}k} - 1}{(1 - \rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\lambda_{\mathbf{J}k}^{q_0}\beta_k.\end{aligned} \tag{A.2}$$

We now have the inputs needed to proof our main results on price and capital value curves. Suppose the price system and the techniques of production described in Section 2.1. Then,

Theorem 1. There cannot be exact quadratic or higher order polynomial price curves. Only exact linear price curves are possible.

Proof. Equation (A.2) indicates that the series $\mathbf{d}_{q_0}$ is a linear combination of $n-1$ linearly independent (eigen)vectors $\mathbf{y}_k \neq \mathbf{0}$, for $k = 2, \ldots, n$. Therefore, a q_0-order price curve polynomial $\mathbf{p}(\rho) = \mathbf{v} + \rho\mathbf{f} + \cdots + \rho^{q_0}\mathbf{f}^{q_0} \iff \mathbf{d}_{q_0} = \mathbf{0} \iff \gamma_2 = \cdots = \gamma_n = 0$. Constraint $\gamma_{k\geq 2} = 0$ for all $\rho \in [0, 1)$ cannot be accomplished by $\lambda_{\mathbf{J}k} - 1 = 0$ (matrix $\mathbf{J}$ is primitive), but only by $\lambda_{\mathbf{J}k}^{q_0}\beta_k = \beta_k \prod_{s=1}^{q_0} \lambda_{\mathbf{J}k} = 0$ for $k = 2, \ldots, n$, which can only happen if $\lambda_{\mathbf{J}k}\beta_k = 0$. But the vectors with the polynomial coefficients $1, 2, \ldots, q_0$ are

$$\begin{aligned}
\mathbf{f}^1 &= \sum\nolimits_{k=2}^{n} \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k\mathbf{x}_k^T}\beta_k\mathbf{y}_k \\
\mathbf{f}^2 &= \sum\nolimits_{k=2}^{n} \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k\mathbf{x}_k^T}\lambda_{\mathbf{J}k}\beta_k\mathbf{y}_k \\
&\vdots \\
\mathbf{f}^{q_0} &= \sum\nolimits_{k=2}^{n} \frac{(\lambda_{\mathbf{J}k} - 1)}{\mathbf{y}_k\mathbf{x}_k^T}\lambda_{\mathbf{J}k}^{q_0-1}\beta_k\mathbf{y}_k.
\end{aligned}$$

Hence, quadratic price curves ($q_0 = 2$) require $\lambda_{\mathbf{J}k}^2\beta_k = 0$ which can only happen if $\lambda_{\mathbf{J}k}\beta_k = 0$ and not all $\beta_{k\geq 2}$ are zero, which in turn implies that $\mathbf{f}^2 = \mathbf{0}$, i.e., price curves are linear. Cubic price curves ($q_0 = 3$) require $\lambda_{\mathbf{J}k}^3\beta_k = 0$ which can only happen if $\lambda_{\mathbf{J}k}\beta_k = 0$ and not all $\beta_{k\geq 2}$ are zero, which in turn implies that $\mathbf{f}^3 = \mathbf{f}^2 = \mathbf{0}$, i.e., price curves are linear; and so on and so forth. Therefore, for a $q_0 \geq 2$, $\mathbf{d}_{q_0} = \mathbf{0} \Rightarrow \mathbf{d}_1 = \mathbf{0} \Rightarrow \mathbf{p}(\rho) = \mathbf{v} + \rho\mathbf{f}$. (Q.E.D.)

Theorem 1 has a similar results on capital value curves.

Corollary 1. There cannot be exact linear or higher order polynomial capital value curves. Only exact constant capital curves are possible.

Proof. Remembering that $\mathbf{y}_k\mathbf{J} = \lambda_{\mathbf{J}k}\mathbf{y}_k$, take (9)-(10) and let us re-express capital value curves as

$$
\begin{aligned}
\mathbf{p}(\rho)\mathbf{J} &= \mathbf{vJ} + \rho\mathbf{g} + \cdots + \rho^{q_0}\mathbf{g}^{q_0} + \mathbf{e}_{q_0}, \\
\mathbf{g}^q &\equiv \mathbf{f}^q\mathbf{J} = \mathbf{f}^{q+1} \\
\mathbf{e}_{q_0} &\equiv \sum\nolimits_{s=q_0+1}^{\infty} \rho^s\mathbf{g}^s = \sum\nolimits_{s=q_0+1}^{\infty} \rho^s\mathbf{f}^s\mathbf{J} = \mathbf{d}_{q_0}\mathbf{J} \\
&= \rho^{q_0+1}\sum_{k=2}^{n}\gamma_k\mathbf{y}_k\mathbf{J} = \rho^{q_0+1}\sum_{k=2}^{n}\gamma_k\lambda_{\mathbf{J}k}\mathbf{y}_k = \rho^{q_0+1}\sum_{k=2}^{n}\delta_k\mathbf{y}_k = \mathbf{d}_{q_0+1} \qquad \text{(A.3)} \\
\delta_k &\equiv \gamma_k\lambda_{\mathbf{J}k} = \frac{\lambda_{\mathbf{J}k} - 1}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\lambda_{\mathbf{J}k}^{q_0+1}\beta_k.
\end{aligned}
$$

Because of (A.3), $\mathbf{e}_{q_0} = \mathbf{d}_{q_0+1}$, for $q_0 = 0, 1, \ldots,$ and the results from Theorem 1 that $\mathbf{d}_2, \mathbf{d}_3, \cdots = \mathbf{0} \Rightarrow \mathbf{d}_1 = \mathbf{0}$ implies that $\mathbf{e}_1, \mathbf{e}_2, \cdots = \mathbf{0} \Rightarrow \mathbf{e}_0 = \mathbf{0} \Rightarrow \mathbf{p}(\rho)\mathbf{J} = \mathbf{vJ}$. (Q.E.D.)

In spite of this, there is plenty of room for polynomial approximations:

Definition 1. A polynomial *approximation* of price curves of order $q_0 \in \{1, 2, \ldots\}$ with precision $\epsilon \in \mathbb{R}^n_+$ is given when $\mathbf{p} = \mathbf{v} + \rho\mathbf{f} + \cdots + \rho^{q_0}\mathbf{f}^{q_0} + \mathbf{d}_{q_0}$ and $0 < \|\mathbf{d}_{q_0}\| \leq \epsilon$.

The smaller ϵ the higher the precision of the polynomial approximation. Now, given (A.2),

$$
\|\mathbf{d}_{q_0}\| \leq \left|\rho^{q_0+1}\right| \sum_{k=2}^{n} |\gamma_k| \, \|\mathbf{y}_k\| = \left|\rho^{q_0+1}\right| \sum_{k=2}^{n} \left|\frac{\lambda_{\mathbf{J}k} - 1}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\right| \left|\lambda_{\mathbf{J}k}^{q_0}\beta_k\right| \|\mathbf{y}_k\|, \qquad \text{(A.4)}
$$

therefore,

Theorem 2. Sufficient conditions to obtain a $q_0 \in \{1, 2, \ldots\}$ order polynomial price curves with precision ϵ are that $\lambda_{\mathbf{J}k}^{q_0}\beta_k$, for $k = 2\ldots, n$, are sufficiently small but not all of them equal to zero.

Proof. Given the assumptions of the price system, the absolute value of subdominant eigenvalues and eigenlabours are bounded by $0 \leq |\lambda_{\mathbf{J}k\geq 2}| < 1$ and $0 \leq |\beta_{k\geq 2}|$, so the limit of (A.4) is

$$
\lim_{\lambda_{\mathbf{J}k}\beta_k \to 0} \|\mathbf{d}_{q_0}\| = \lim_{\lambda_{\mathbf{J}k}\beta_k \to 0} \left|\rho^{q_0+1}\right| \sum\nolimits_{k=2}^{n} \left|\frac{\lambda_{\mathbf{J}k} - 1}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\right| \left|\lambda_{\mathbf{J}k}^{q_0-1}\lambda_{\mathbf{J}k}\beta_k\right| \|\mathbf{y}_k\| = 0.
$$

Hence, for any positive ϵ, it is possible to find a technique of production with *sufficiently* small $\lambda_{\mathbf{J}k}$ and/or β_k such that $0 < \|\mathbf{d}_{q_0}\| \leq \epsilon$. (Q.E.D.)

Remark. These are *sufficient* conditions because the terms $\left|\frac{\lambda_{\mathbf{J}k}-1}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\right|$ and $\|\mathbf{y}_k\|$ can also improve or worsen the approximation precision.

Definition 2. A polynomial approximation of capital value curves of order $q_0 \in \{0, 1, \dots\}$ with precision $\epsilon \in \mathbb{R}^n_+$ is given when $\mathbf{p}(\rho)\mathbf{J} = \mathbf{vJ} + \rho\mathbf{g} + \cdots + \rho^{q_0}\mathbf{g}^{q_0} + \mathbf{e}_{q_0}$ and $0 < \|\mathbf{e}_{q_0}\| \leq \epsilon$.

Corollary 2. Sufficient conditions to obtain a $q_0 \in \{0, 1, \dots\}$ order polynomial capital value curves with accuracy ϵ are that $\lambda_{\mathbf{J}k}^{q_0+1}\beta_k$ are sufficiently small but not all of them equal to zero.

Equation (A.3) tells us that $\mathbf{e}_{q_0} = \mathbf{d}_{q_0+1}$ and (A.4) implies that

$$\|\mathbf{d}_{q_0+1}\| = \|\mathbf{e}_{q_0}\| \leq \left|\rho^{q_0+1}\right| \sum\nolimits_{k=2}^{n} \left|\frac{\lambda_{\mathbf{J}k} - 1}{(1 - \rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\right| \left|\lambda_{\mathbf{J}k}^{(q_0+1)}\beta_k\right| \|\mathbf{y}_k\| .$$

If the technique of production is characterised by a sufficiently small $\lambda_{\mathbf{J}k}^{(q_0+1)}\beta_k$ such that $0 < \|\mathbf{d}_{q_0+1}\| \leq \epsilon$, then $0 < \|\mathbf{e}_{q_0}\| \leq \epsilon$. Hence, approximate linear price curves $(q_0 + 1) = 1$ implies approximately constant capital value curves $q_0 = 0$; approximate quadratic price curves $(q_0 + 1) = 2$ implies approximately non-zero sloped linear capital value curves $q_0 = 1$; etc. (Q.E.D.)

B Database construction

B.1 The WIOD database

The *World Input-Output Database* (WIOD), in its 2016 release, constructed by Timmer et al. (2015), provides estimates of annual time-series of input–output tables (IOTs) covering 43 countries — 28 EU countries and 15 other major countries in the world — for the period from 2000 to 2014. The country sample represents up to 86% of the world economy in 2016 and includes rich- and middle-income economies of diverse industrial structures and development profiles — least developed countries are not included in the sample. The list of the 43 economies, and their acronym used throughout the document (in parenthesis), is given in table B.1.

Australia (AUS) [49]	Austria (AUT) [54]	Belgium (BEL) [54]	Bulgaria (BGR) [54]
Brazil (BRA) [47]	Canada (CAN) [51]	Switzerland (CHE) [49]	China, People's Republic of (CHN) [47]
Cyprus (CYP) [54]	Czech Republic (CZE) [54]	Germany (DEU) [54]	Denmark (DNK) [54]
Spain (ESP) [54]	Estonia (EST) [54]	Finland (FIN) [54]	France (FRA) [54]
United Kingdom of Great Britain and Northern Ireland (GBR) [54]	Greece (GRC) [54]	Croatia (HRV) [54]	Hungary (HUN) [54]
Indonesia (IDN) [47]	India (IND) [45]	Ireland (IRL) [54]	Italy (ITA) [54]
Japan (JPN) [50]	Republic of Korea (KOR) [53]	Lithuania (LTU) [54]	Luxembourg (LUX) [52]
Latvia (LVA) [54]	Mexico (MEX) [52]	Malta (MLT) [52]	Netherlands (NLD) [54]
Norway (NOR) [54]	Poland (POL) [54]	Portugal (PRT) [54]	Romania (ROU) [54]
Russian Federation (RUS) [33]	Slovakia (SVK) [54]	Slovenia (SVN) [54]	Sweden (SWE) [53]
Turkey (TUR) [46]	Taiwan (TWN) [54]	United States (USA) [54]	

Table B.1: List of the 43 countries included in the WIOD database, 2016 release, and number of industries for which there is information available in each country-year.

The base for the construction of the IOTs are the supply and use tables (SUTs), which are obtained from official national sources and are adapted to a 56 industries common disaggregation detail based on the 2008 System of National Accounts (SNA2008) framework. The input years and the number of releases for which SUTs are available are uneven and dispersed with the base methodology drawing from the SNA2008, SNA1993, and International System of Industrial Classification, Revision 3, frameworks. From this information, world industry-by-industry IOTs are constructed from which we obtain the IOTs for each country-year. The WIOD includes fictitious industries which are statistical artifacts to balance the tables. Hence, we decide to omit industries T ("Activities of households as employers; undifferentiated goods- and services- producing activities of households for own use") and U ("Activities of extraterritorial organizations and bodies"), whose entries are mostly zeros. Table B.2 gives the list of the final 54 industries.

ISIC4 code	Sector description
A01	Crop and animal production, hunting and related service activities
A02	Forestry and logging
A03	Fishing and aquaculture
B	Mining and quarrying
C10_C12	Manufacture of food products, beverages and tobacco products
C13_C15	Manufacture of textiles, wearing apparel and leather products
C16	Manufacture of wood and of products of wood and cork, except furniture; straw and plaiting materials
C17	Manufacture of paper and paper products
C18	Printing and reproduction of recorded media
C19	Manufacture of coke and refined petroleum products
C20	Manufacture of chemicals and chemical products
C21	Manufacture of basic pharmaceutical products and pharmaceutical preparations
C22	Manufacture of rubber and plastic products
C23	Manufacture of other non-metallic mineral products
C24	Manufacture of basic metals
C25	Manufacture of fabricated metal products, except machinery and equipment
C26	Manufacture of computer, electronic and optical products
C27	Manufacture of electrical equipment
C28	Manufacture of machinery and equipment n.e.c.
C29	Manufacture of motor vehicles, trailers and semi-trailers
C30	Manufacture of other transport equipment
C31_C32	Manufacture of furniture; other manufacturing
C33	Repair and installation of machinery and equipment
D35	Electricity, gas, steam and air conditioning supply
E36	Water collection, treatment and supply
E37_E39	Sewerage; waste collection, treatment and disposal activities; materials recovery; other waste services
F	Construction
G45	Wholesale and retail trade and repair of motor vehicles and motorcycles
G46	Wholesale trade, except of motor vehicles and motorcycles
G47	Retail trade, except of motor vehicles and motorcycles
H49	Land transport and transport via pipelines
H50	Water transport
H51	Air transport
H52	Warehousing and support activities for transportation
H53	Postal and courier activities
I	Accommodation and food service activities
J58	Publishing activities
J59_J60	Motion picture, video and television production, sound and music; programming and broadcasting
J61	Telecommunications
J62_J63	Computer programming, consultancy and related activities; information service activities
K64	Financial service activities, except insurance and pension funding
K65	Insurance, reinsurance and pension funding, except compulsory social security
K66	Activities auxiliary to financial services and insurance activities
L68	Real estate activities
M69_M70	Legal and accounting activities; activities of head offices; management consultancy activities
M71	Architectural and engineering activities; technical testing and analysis
M72	Scientific research and development
M73	Advertising and market research
M74_M75	Other professional, scientific and technical activities; veterinary activities
N	Administrative and support service activities
O84	Public administration and defence; compulsory social security
P85	Education
Q	Human health and social work activities
R_S	Other service activities

Table B.2: List with the 54 industries from the WIOD, 2016 release, considered in the sample. *Notes:* Omitted industries are "Activities of households as employers; undifferentiated goods- and services- producing activities of households for own use" (T) and "Activities of extraterritorial organisations and bodies" (U).

Not every country in the WIOD has information for the 54 industries. The number of industries with information for each country is given in squared brackets in Table B.1. The country with less information is Russia (with 31 industries), but most of the economies fluctuate between 52 and 54 industries.

The WIOD provides in addition socio-economic accounts (SEAs) containing

Symbol	Variable	Units
GO	Gross output by industry at current basic prices	in millions of national currency
II	Intermediate inputs at current purchasers' prices	in millions of national currency
VA	Gross value added at current basic prices	in millions of national currency
EMP	Number of persons engaged	thousands
EMPE	Number of employees	thousands
H_EMPE	Total hours worked by employees	millions
COMP	Compensation of employees	in millions of national currency
LAB	Labour compensation	in millions of national currency
CAP	Capital compensation	in millions of national currency
K	Nominal capital stock	in millions of national currency
GO_PI	Price levels gross output	2010=100
II_PI	Price levels of intermediate inputs	2010=100
VA_PI	Price levels of gross value added	2010=100
GO_QI	Gross output, volume indices	2010=100
II_QI	Intermediate inputs, volume indices	2010=100
VA_QI	Gross value added, volume indices	2010=100

Table B.3: List of the variables in the Socio-Economic Accounts in the WIOD database, 2016 release, and their description.

industry-level data, under the same industry classification system as the WIOTs, on the uses of primary inputs (capital and labour), intermediate inputs, gross output, and the components of value added at current and constant prices. Table B.3 provides the full description of the available information. A comprehensive overview of the sources and methodological choices for the original release can be found in Dietzenbacher et al. (2013).

The information for the national and international industry-by-industry and supply and use tables corresponds to current market international dollar prices. The value data of the SAEs are denoted in millions of national currency. Values were converted into dollars using the exchange rates provided in the WIOD as an independent file.

B.2 Construction of the input-coefficient matrix and the labour coefficient vector from the WIOD

Empirical computations of production-price models cannot be done using as parameters the techniques of production. In practice, what is used is the information from the economies' productive structure.[B.1] Its first component is the intermediate inputs cost-shares matrix or Leontief's technical coefficient matrix. This matrix is constructed with the information of the $n \times n$ table with interindustry intermediate money valued flows between industries, $\mathbf{Z} \equiv [z_{ij}]$ and the vector with the money

[B.1] See Ochoa (1984, appendices), Chilcote (1997, appendices), Shaikh (2012, Data appendix), Mariolis and Tsoulfidis (2016b, ch. 2), and Torres-González (2020, Appendix B) for detailed discussions on the construction of databases to conduct empirical computations of production-price models.

value of gross output, $\mathbf{x} \equiv [x_j]$: $\mathbf{Z}\hat{\mathbf{x}}^{-1} = [\frac{z_{ij}}{x_j}] \equiv \mathbf{A} \equiv [a_{ij}]$. The second component is the vector with the labour coefficients. This vector is constructed from some estimate of the labour input in each industry $\mathbf{L} \equiv [L_j]$ divided by the gross output of that industry: $\mathbf{L}\hat{\mathbf{x}}^{-1} \equiv \left[\frac{L_j}{x_j}\right] \equiv \mathbf{l} \equiv [l_j]$.

Interindustry flows of intermediate inputs z_{ij}, for $i, j = 1, \ldots, n$, can involve transaction between domestic industries z_{ij}^D and imports of domestic industries from foreign industries z_{ij}^M. In the construction of Leontief's technical coefficient matrices $\mathbf{A}$ we aggregate the domestic and imported flows: $z_{ij} = z_{ij}^D + z_{ij}^M$. This means that we abstract from structural differences in the input-output relations that exists between domestic and foreign industries. Therefore, for the construction of matrices $\mathbf{Z}$ for each country-year we take the domestic table $\mathbf{Z}^D$ and add them the tables with imported interindustry money flows of intermediate inputs $\mathbf{Z}^M$ from the remaining 42 countries and the rest of the world estimate.

The constructed database uses two measures of the labour input L_j. For Section 3.2, Appendix D.1, and Section 4, we construct the skilled-adjusted labour vector using the labour compensation of persons engaged in production (LAB) and divide it by the economy-wide average wage rate, $\frac{\sum_{j=1}^{54} LAB_j}{\sum_{j=1}^{54} EMP_j}$. With this, industries' average wage rate differentials serve as skill indices which weight the number of persons engaged in production in each industry.[B.2] As for Appendix D.2, we use as L_j the number of persons engaged in production (EMP) from the SAE. The latter variant is used as a measure of robustness of the results.

Finally, for each of the 645 economies (43 countries and 15 years) the dimension of the system in the empirical model is $n = 54$. We remind that there is no information for all these 54 industries for the 43 countries. Therefore, some columns/rows of matrix $\mathbf{A}$ and entries in vector $\mathbf{l}$ will be filled with zeroes. This feature has the effect that the variates we compute will have some "artificial" zeroes. However, this outcome will not bias our results. The statistical properties of the variates, as we shall see in sections 3 and 4 and appendices C and D, are robust to these artificial zeroes. On average, in every year there are 52.04 industries per countries, so there are 2,271 observations per year.

[B.2] See Ochoa (1984, Appendix B.2) ad (Shaikh, 2012, p. 98) for two alternative approaches to construct a skill-adjusted labour vector. EMP equals employees "EMPE" plus self-employed.

C Detailed empirical evidence - Skill-Adjusted labour Vector ($\frac{1}{w}LAB_j$)

The following figures and tables expand the selected results presented in Section 3.2.

C.1 Complex plane

Figure C.1. Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for AUS, AUT, BEL, BGR, BRA, CAN, CHE, CHN, CYP, CZE, DEU, DNK, ESP, and EST.

Figure C.2. Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for FIN, FRA, GBR, GRC, HRV, HUN, IDN, IND, IRL, ITA, JPN, KOR, LTU, and LUX.

Figure C.3. Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for LVA, MEX, MLT, NLD, NOR, POL, PRT, ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

C.2 Boxplots

Figure C.4. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for AUS, AUT, BEL, BGR, BRA, CAN, and CHE.

Figure C.5. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for CHN, CYP, CZE, DEU, DNK, ESP, and EST.

Figure C.6. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for FIN, FRA, GBR, GRC, HRV, HUN, and IDN.

Figure C.7. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for IND, IRL, ITA, JPN, KOR, LTU, and LUX.

Figure C.8. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, for LVA, MEX, MLT, NLD, NOR, POL, and PRT.

Figure C.9. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

C.3 Descriptive statistics

Table C.1. Descriptive statistics of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ for the year 2000, full WIOD sample.

Table C.2. Descriptive statistics of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ for the year 2014, full WIOD sample.

C.4 Rankplots

Figure C.10. Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 54 industries, 2000–2014, for AUS, AUT, BEL, BGR, BRA, CAN, CHE, CHN, CYP, CZE, DEU, DNK, ESP, EST, FIN, FRA, GBR, GRC, HRV, HUN, IDN, IND, IRL, and ITA.

Figure C.11. Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 54 industries, 2000–2014, for JPN, KOR, LTU, LUX, LVA, MEX, MLT, NLD, NOR, POL, PRT, ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

C.5 Deviations from proportionality

Figure C.12. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix $\mathbf{J}$, 2000-2014, 54 industries, for AUS, AUT, BEL, BGR, BRA, CAN, CHE, CHN, CYP, CZE, and DEU.

Figure C.13. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix $\mathbf{J}$, 2000-2014, 54 industries, for DNK, ESP, EST, FIN, FRA, GBR, GRC, HRV, HUN, IDN, and IND.

Figure C.14. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix $\mathbf{J}$, 2000-2014, 54 industries, for IRL, ITA, JPN, KOR, LTU, LUX, LVA, MEX, MLT, and NLD.

Figure C.15. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix **J**, 2000-2014, 54 industries, for POL, PRT, ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

Figure C.1: Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

Figure C.2: Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

Figure C.3: Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

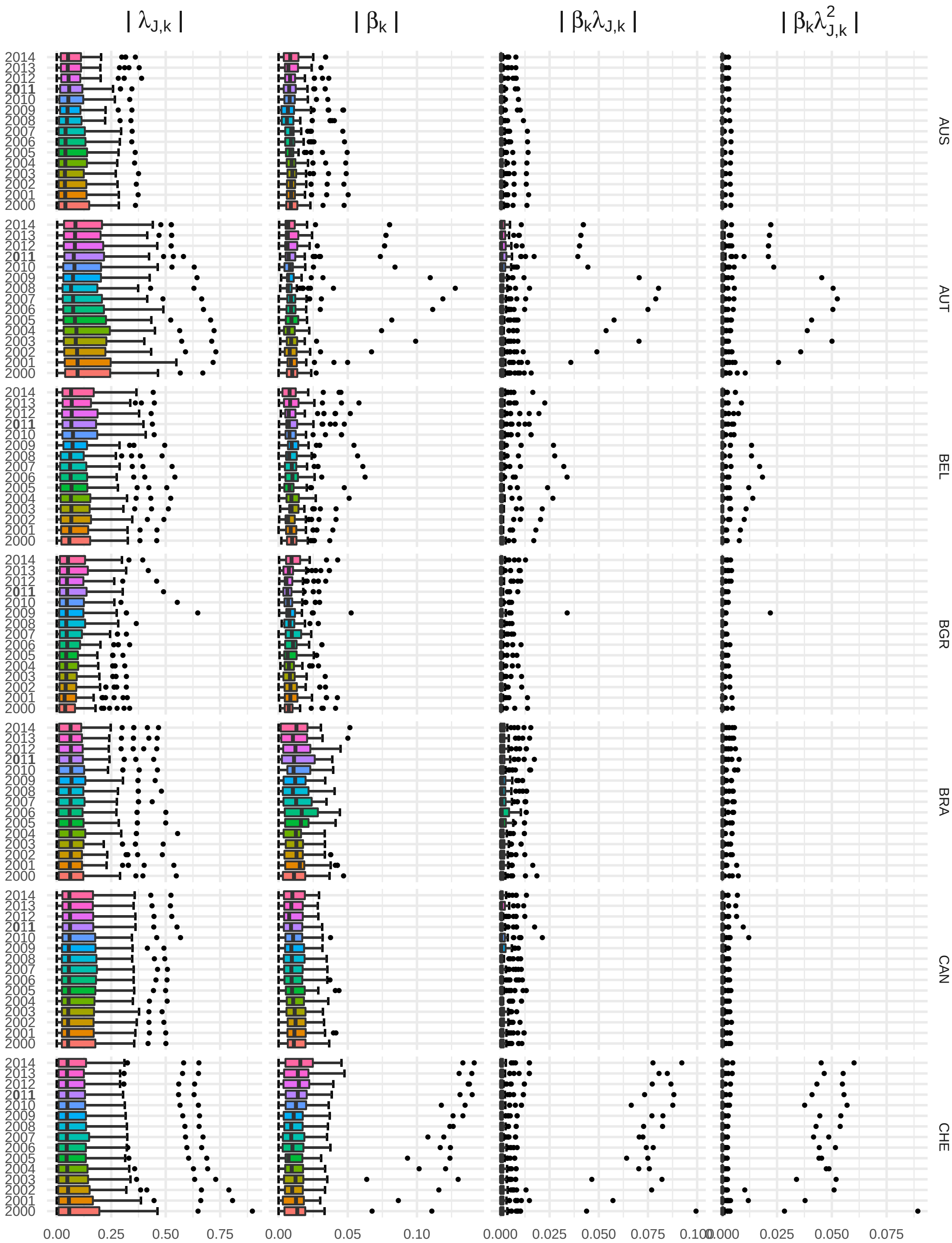


Figure C.4: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

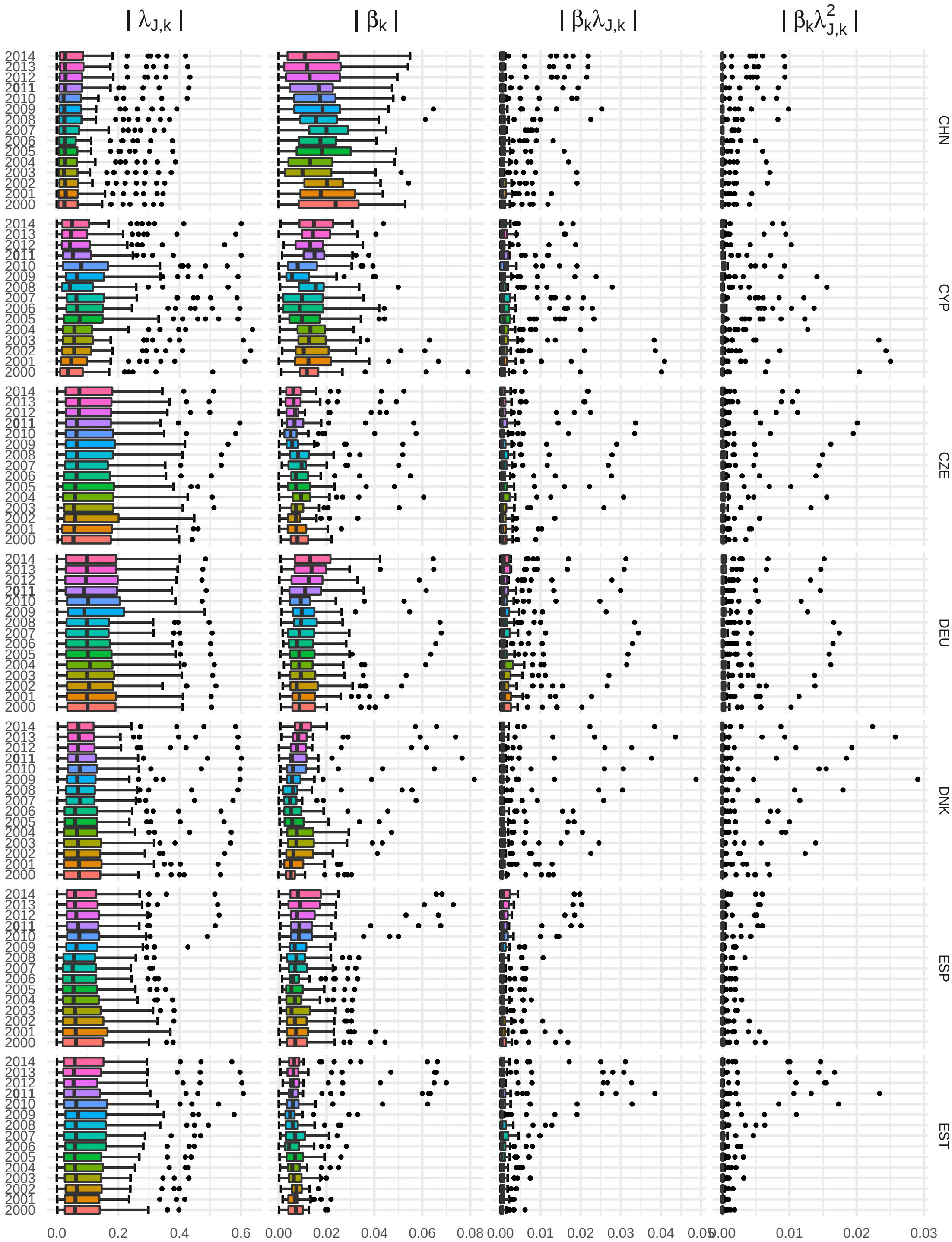


Figure C.5: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

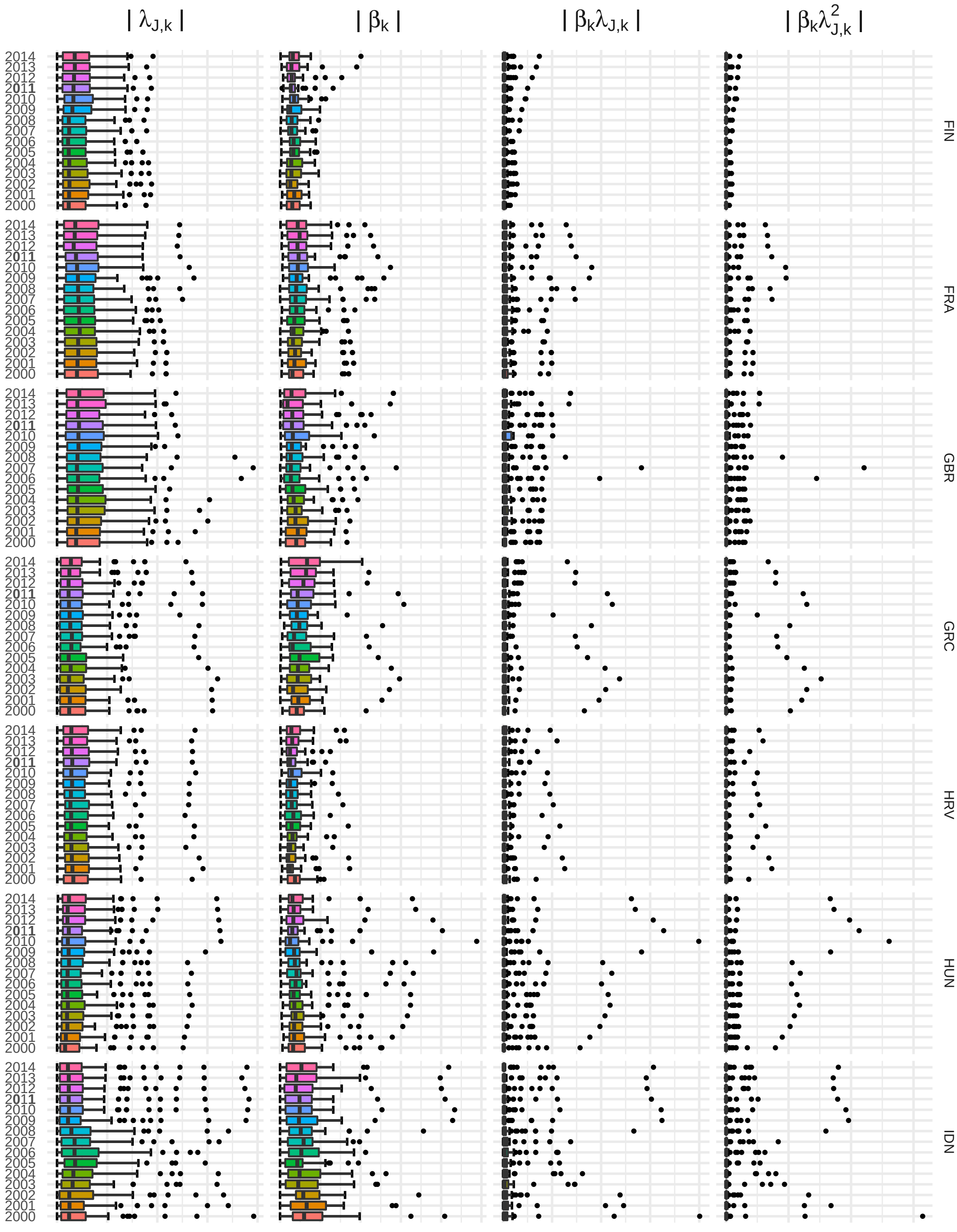


Figure C.6: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

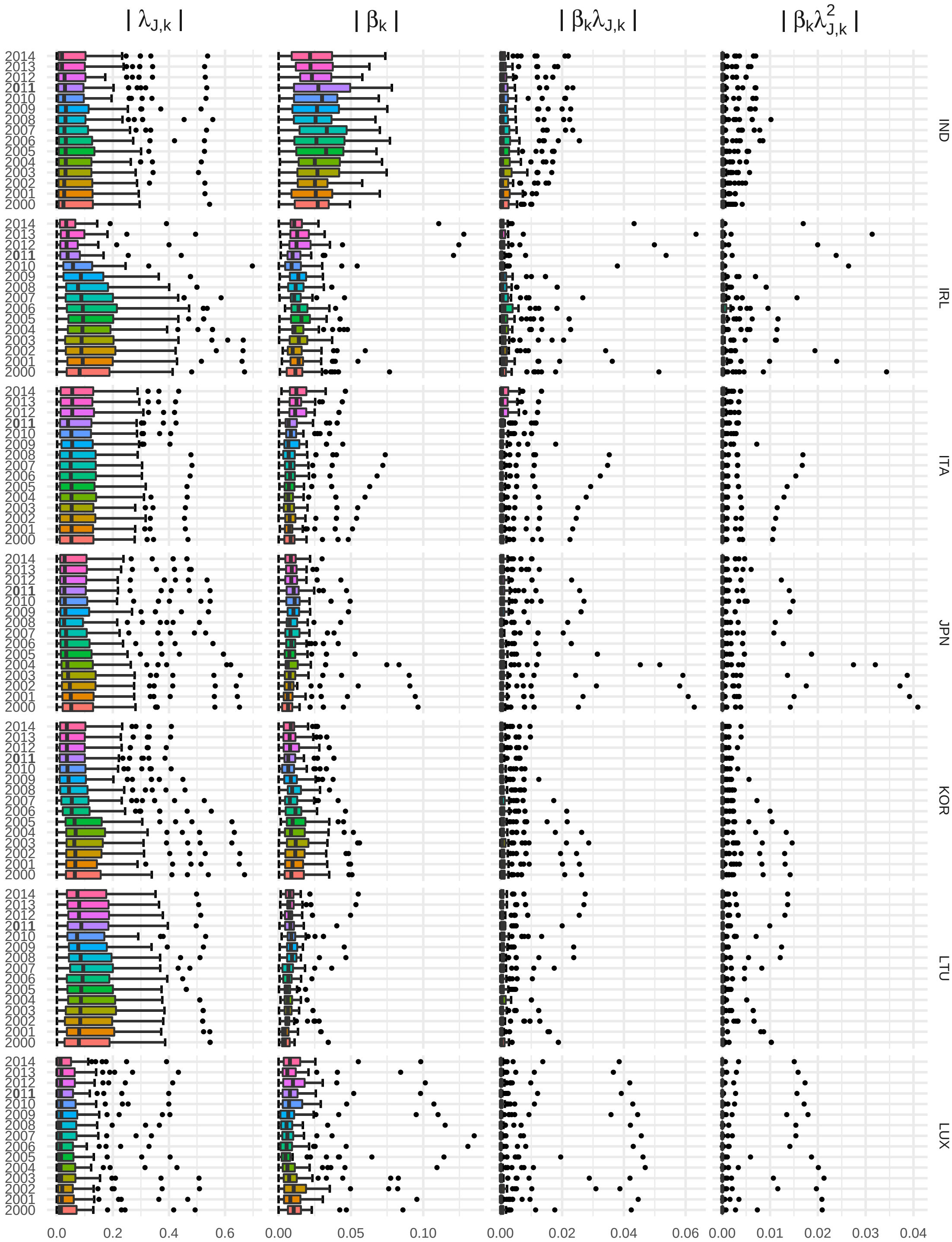


Figure C.7: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

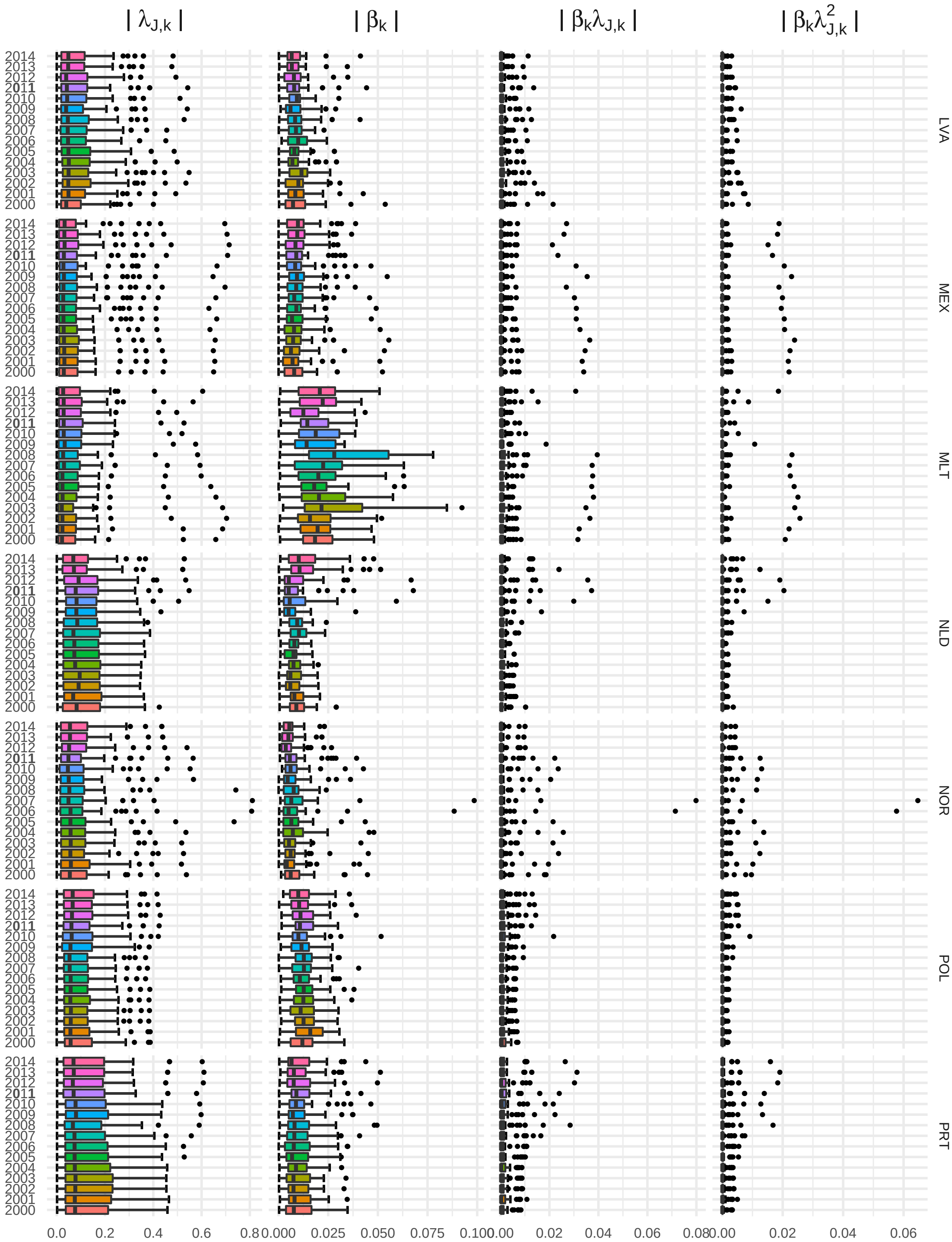


Figure C.8: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

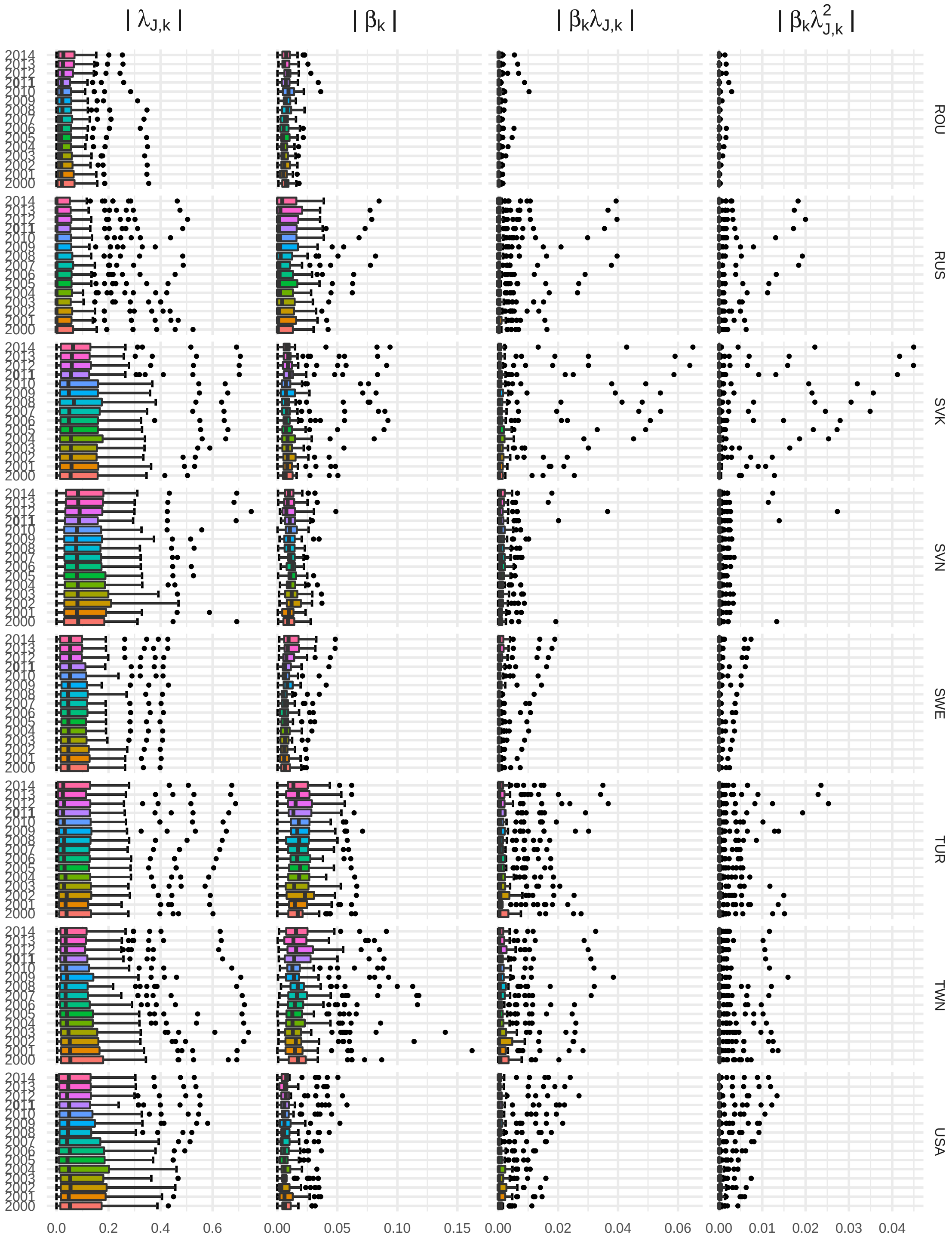


Figure C.9: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

Table C.1: Descriptive statistics for 43 countries, year 2000[1,2]

Country	$\lvert\lambda_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda^2_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$
AUS	0.040	0.140	0.359	1	0.361	0.285	0.009	0.008	0.026	0.235	0.201	0.137	0.000	0.001	0.003	0.235	0.057	0.027	0	0.000	0.001	0.235	0.016	0.005
AUT	0.101	0.217	0.582	1	0.670	0.567	0.010	0.008	0.025	0.165	0.165	0.143	0.001	0.002	0.005	0.165	0.095	0.074	0	0.000	0.001	0.165	0.064	0.042
BEL	0.061	0.142	0.376	1	0.459	0.382	0.010	0.009	0.029	0.210	0.176	0.123	0.000	0.001	0.003	0.210	0.081	0.033	0	0.000	0.001	0.210	0.037	0.011
BGR	0.039	0.084	0.222	1	0.334	0.309	0.008	0.006	0.020	0.151	0.272	0.213	0.000	0.000	0.001	0.151	0.091	0.048	0	0.000	0.000	0.151	0.030	0.015
BRA	0.062	0.111	0.288	1	0.548	0.395	0.011	0.016	0.043	0.166	0.284	0.222	0.001	0.002	0.004	0.166	0.112	0.077	0	0.000	0.000	0.166	0.044	0.028
CAN	0.058	0.157	0.416	1	0.501	0.419	0.011	0.013	0.039	0.211	0.174	0.158	0.001	0.002	0.004	0.211	0.052	0.044	0	0.001	0.001	0.211	0.018	0.015
CHE	0.064	0.187	0.476	1	0.898	0.648	0.014	0.015	0.042	0.179	0.620	0.378	0.001	0.003	0.007	0.179	0.556	0.245	0	0.001	0.002	0.179	0.499	0.159
CHN	0.025	0.062	0.161	1	0.341	0.315	0.024	0.026	0.073	0.221	0.239	0.219	0.001	0.001	0.003	0.221	0.054	0.037	0	0.000	0.000	0.221	0.018	0.010
CYP	0.037	0.075	0.199	1	0.508	0.323	0.012	0.008	0.028	0.054	1.460	1.139	0.000	0.001	0.003	0.054	0.742	0.368	0	0.000	0.000	0.054	0.377	0.119
CZE	0.055	0.170	0.447	1	0.441	0.398	0.008	0.007	0.023	0.210	0.105	0.097	0.001	0.001	0.003	0.210	0.042	0.020	0	0.000	0.001	0.210	0.017	0.009
DEU	0.102	0.171	0.464	1	0.504	0.409	0.009	0.009	0.029	0.185	0.219	0.218	0.001	0.002	0.006	0.185	0.110	0.076	0	0.000	0.001	0.185	0.055	0.031
DNK	0.073	0.124	0.336	1	0.533	0.415	0.005	0.004	0.014	0.196	0.155	0.147	0.000	0.001	0.002	0.196	0.067	0.062	0	0.000	0.000	0.196	0.036	0.025
ESP	0.063	0.146	0.386	1	0.379	0.358	0.007	0.008	0.025	0.199	0.224	0.193	0.000	0.001	0.003	0.199	0.085	0.069	0	0.000	0.001	0.199	0.032	0.025
EST	0.061	0.119	0.323	1	0.398	0.361	0.007	0.006	0.019	0.172	0.120	0.114	0.000	0.001	0.002	0.172	0.036	0.020	0	0.000	0.000	0.172	0.011	0.007
FIN	0.048	0.090	0.249	1	0.355	0.272	0.008	0.008	0.025	0.187	0.101	0.092	0.000	0.001	0.002	0.187	0.016	0.015	0	0.000	0.000	0.187	0.006	0.003
FRA	0.086	0.133	0.364	1	0.435	0.380	0.008	0.009	0.028	0.200	0.213	0.200	0.001	0.002	0.005	0.200	0.093	0.076	0	0.000	0.001	0.200	0.040	0.029
GBR	0.083	0.157	0.432	1	0.483	0.438	0.010	0.012	0.034	0.189	0.220	0.167	0.001	0.001	0.002	0.189	0.079	0.073	0	0.000	0.000	0.189	0.032	0.029
GRC	0.051	0.101	0.264	1	0.621	0.348	0.010	0.009	0.028	0.131	0.409	0.211	0.000	0.001	0.002	0.131	0.254	0.038	0	0.000	0.000	0.131	0.158	0.013
HRV	0.066	0.114	0.312	1	0.539	0.335	0.009	0.008	0.024	0.133	0.205	0.188	0.001	0.002	0.004	0.133	0.101	0.044	0	0.000	0.001	0.133	0.055	0.011
HUN	0.032	0.078	0.208	1	0.503	0.385	0.008	0.011	0.032	0.159	0.396	0.392	0.000	0.001	0.001	0.159	0.197	0.111	0	0.000	0.000	0.159	0.099	0.043
IDN	0.049	0.097	0.260	1	0.785	0.558	0.015	0.019	0.055	0.158	0.646	0.513	0.001	0.002	0.004	0.158	0.507	0.286	0	0.000	0.001	0.158	0.398	0.160
IND	0.029	0.129	0.329	1	0.545	0.296	0.027	0.023	0.070	0.155	0.318	0.287	0.001	0.003	0.007	0.155	0.065	0.063	0	0.000	0.001	0.155	0.027	0.018
IRL	0.082	0.163	0.442	1	0.669	0.481	0.012	0.011	0.032	0.120	0.640	0.345	0.001	0.002	0.004	0.120	0.428	0.149	0	0.000	0.001	0.120	0.287	0.072
ITA	0.052	0.127	0.329	1	0.468	0.345	0.008	0.007	0.021	0.195	0.247	0.211	0.000	0.001	0.002	0.195	0.116	0.068	0	0.000	0.000	0.195	0.054	0.022
JPN	0.056	0.125	0.333	1	0.651	0.564	0.007	0.008	0.022	0.178	0.541	0.251	0.000	0.001	0.002	0.178	0.352	0.142	0	0.000	0.000	0.178	0.229	0.080
KOR	0.064	0.134	0.369	1	0.670	0.540	0.009	0.015	0.042	0.111	0.457	0.438	0.001	0.002	0.005	0.111	0.237	0.188	0	0.000	0.001	0.111	0.128	0.077
LTU	0.079	0.160	0.429	1	0.547	0.386	0.004	0.005	0.016	0.134	0.256	0.082	0.000	0.001	0.003	0.134	0.140	0.028	0	0.000	0.001	0.134	0.076	0.011
LUX	0.017	0.067	0.174	1	0.493	0.417	0.011	0.010	0.032	0.056	1.543	0.841	0.000	0.001	0.002	0.056	0.761	0.316	0	0.000	0.000	0.056	0.376	0.132
LVA	0.039	0.093	0.249	1	0.400	0.304	0.008	0.010	0.030	0.078	0.692	0.470	0.000	0.001	0.003	0.078	0.277	0.143	0	0.000	0.000	0.078	0.111	0.044
MEX	0.030	0.078	0.207	1	0.650	0.441	0.008	0.009	0.025	0.087	0.599	0.338	0.000	0.000	0.001	0.087	0.389	0.087	0	0.000	0.000	0.087	0.253	0.033
MLT	0.022	0.070	0.182	1	0.659	0.525	0.020	0.015	0.049	0.100	0.480	0.430	0.000	0.001	0.003	0.100	0.316	0.084	0	0.000	0.000	0.100	0.208	0.022
NLD	0.082	0.154	0.410	1	0.425	0.365	0.009	0.008	0.026	0.199	0.146	0.097	0.000	0.001	0.004	0.199	0.052	0.022	0	0.000	0.001	0.199	0.018	0.009
NOR	0.060	0.114	0.304	1	0.537	0.416	0.006	0.008	0.023	0.174	0.257	0.192	0.000	0.001	0.002	0.174	0.107	0.103	0	0.000	0.000	0.174	0.055	0.045
POL	0.057	0.112	0.314	1	0.387	0.380	0.012	0.011	0.033	0.203	0.163	0.134	0.001	0.002	0.005	0.203	0.035	0.033	0	0.000	0.001	0.203	0.009	0.008
PRT	0.079	0.193	0.521	1	0.459	0.441	0.008	0.013	0.037	0.175	0.198	0.139	0.000	0.001	0.003	0.175	0.048	0.044	0	0.000	0.001	0.175	0.019	0.019
ROU	0.021	0.065	0.166	1	0.354	0.185	0.007	0.005	0.018	0.153	0.118	0.103	0.000	0.001	0.001	0.153	0.010	0.010	0	0.000	0.000	0.153	0.002	0.001
RUS	0.000	0.064	0.159	1	0.525	0.456	0.001	0.015	0.037	0.213	0.198	0.142	0.000	0.001	0.002	0.213	0.076	0.032	0	0.000	0.000	0.213	0.029	0.009
SVK	0.054	0.144	0.373	1	0.504	0.416	0.007	0.008	0.025	0.162	0.312	0.268	0.000	0.001	0.004	0.162	0.157	0.093	0	0.000	0.001	0.162	0.079	0.032
SVN	0.085	0.182	0.485	1	0.693	0.448	0.009	0.010	0.029	0.203	0.137	0.109	0.001	0.001	0.003	0.203	0.095	0.034	0	0.000	0.001	0.203	0.066	0.011
SWE	0.046	0.105	0.279	1	0.398	0.334	0.006	0.007	0.022	0.217	0.111	0.102	0.000	0.000	0.001	0.217	0.034	0.019	0	0.000	0.000	0.217	0.011	0.008
TUR	0.039	0.128	0.330	1	0.601	0.469	0.017	0.012	0.039	0.169	0.384	0.364	0.000	0.004	0.011	0.169	0.163	0.148	0	0.001	0.002	0.169	0.089	0.073
TWN	0.060	0.162	0.423	1	0.693	0.660	0.017	0.014	0.044	0.049	1.786	1.492	0.001	0.003	0.009	0.049	0.416	0.256	0	0.000	0.001	0.049	0.152	0.135
USA	0.053	0.175	0.453	1	0.430	0.388	0.006	0.007	0.022	0.191	0.166	0.152	0.000	0.001	0.004	0.191	0.059	0.029	0	0.000	0.001	0.191	0.023	0.010
Mean	0.055	0.126	0.335	1	0.520	0.408	0.010	0.010	0.031	0.164	0.294	0.225	0.000	0.001	0.004	0.164	0.142	0.075	0	0.000	0.001	0.164	0.081	0.032
SD	0.023	0.040	0.106	0	0.128	0.098	0.005	0.004	0.013	0.047	0.377	0.276	0.000	0.001	0.002	0.047	0.188	0.087	0	0.000	0.000	0.047	0.123	0.043

[1] The table reports the median, the interquartile range (IQR), the outlier frontier, the first value ($\lambda_{J,1}, \beta_1, \beta_1\lambda_{J,1}, \beta_1\lambda^2_{J,1}$), the maximum value different from $k = 1$, and the maximum subdominant value normalised by the corresponding $k = 1$.
Source: Authors' calculations based on WIOD tables, 54 industries.

Table C.2: Descriptive statistics for 43 countries, year 2014[1,2]

Country	$\lvert\lambda_{J,k}\rvert$						$\lvert\beta_k\rvert$						$\lvert\beta_k\lambda_{J,k}\rvert$						$\lvert\beta_k\lambda^2_{J,k}\rvert$					
	Median	IQR	RW	$k=1$	Max $k\geq2/k{=}1$	Sub $k\geq2/k{=}1$	Median	IQR	RW	$k=1$	Max $k\geq2/k{=}1$	Sub $k\geq2/k{=}1$	Median	IQR	RW	$k=1$	Max $k\geq2/k{=}1$	Sub $k\geq2/k{=}1$	Median	IQR	RW	$k=1$	Max $k\geq2/k{=}1$	Sub $k\geq2/k{=}1$
AUS	0.052	0.092	0.250	1	0.361	0.316	0.008	0.010	0.030	0.207	0.164	0.121	0.000	0.001	0.002	0.207	0.038	0.020	0	0.000	0.000	0.207	0.014	0.006
AUT	0.086	0.174	0.468	1	0.526	0.477	0.007	0.007	0.022	0.185	0.433	0.144	0.000	0.002	0.005	0.185	0.227	0.057	0	0.001	0.001	0.185	0.119	0.023
BEL	0.066	0.143	0.385	1	0.443	0.366	0.008	0.010	0.027	0.218	0.206	0.200	0.000	0.001	0.004	0.218	0.075	0.031	0	0.000	0.001	0.218	0.028	0.011
BGR	0.052	0.127	0.332	1	0.395	0.331	0.010	0.010	0.031	0.153	0.280	0.227	0.000	0.001	0.002	0.153	0.084	0.061	0	0.000	0.000	0.153	0.025	0.017
BRA	0.063	0.120	0.312	1	0.467	0.416	0.013	0.019	0.050	0.160	0.323	0.192	0.001	0.002	0.005	0.160	0.097	0.075	0	0.000	0.001	0.160	0.035	0.029
CAN	0.062	0.153	0.410	1	0.524	0.432	0.010	0.015	0.042	0.190	0.154	0.138	0.000	0.001	0.003	0.190	0.070	0.042	0	0.000	0.000	0.190	0.037	0.014
CHE	0.051	0.149	0.382	1	0.652	0.582	0.016	0.021	0.057	0.145	0.977	0.920	0.001	0.001	0.003	0.145	0.637	0.536	0	0.000	0.000	0.145	0.416	0.312
CHN	0.029	0.079	0.209	1	0.420	0.358	0.011	0.021	0.057	0.208	0.264	0.257	0.000	0.001	0.002	0.208	0.105	0.087	0	0.000	0.000	0.208	0.044	0.029
CYP	0.050	0.097	0.262	1	0.601	0.414	0.015	0.014	0.044	0.083	0.527	0.371	0.001	0.001	0.004	0.083	0.218	0.182	0	0.000	0.001	0.083	0.109	0.090
CZE	0.075	0.153	0.411	1	0.511	0.414	0.006	0.006	0.018	0.225	0.232	0.190	0.000	0.001	0.004	0.225	0.097	0.096	0	0.000	0.001	0.225	0.050	0.040
DEU	0.102	0.162	0.436	1	0.485	0.401	0.014	0.016	0.047	0.191	0.338	0.222	0.001	0.002	0.006	0.191	0.164	0.089	0	0.000	0.001	0.191	0.080	0.036
DNK	0.072	0.102	0.290	1	0.582	0.479	0.009	0.007	0.023	0.202	0.327	0.283	0.001	0.001	0.002	0.202	0.190	0.111	0	0.000	0.000	0.202	0.111	0.043
ESP	0.064	0.103	0.292	1	0.515	0.357	0.008	0.013	0.037	0.161	0.424	0.409	0.001	0.002	0.006	0.161	0.123	0.115	0	0.000	0.001	0.161	0.037	0.031
EST	0.058	0.129	0.346	1	0.570	0.470	0.006	0.005	0.017	0.181	0.367	0.344	0.000	0.001	0.002	0.181	0.172	0.138	0	0.000	0.000	0.181	0.081	0.056
FIN	0.073	0.109	0.297	1	0.383	0.295	0.008	0.006	0.021	0.217	0.231	0.088	0.001	0.001	0.002	0.217	0.068	0.020	0	0.000	0.000	0.217	0.020	0.005
FRA	0.080	0.135	0.368	1	0.489	0.360	0.012	0.013	0.036	0.185	0.285	0.231	0.001	0.001	0.003	0.185	0.139	0.083	0	0.000	0.000	0.185	0.068	0.030
GBR	0.093	0.151	0.417	1	0.475	0.391	0.007	0.014	0.037	0.181	0.388	0.210	0.001	0.001	0.003	0.181	0.152	0.065	0	0.000	0.001	0.181	0.059	0.031
GRC	0.059	0.085	0.226	1	0.515	0.350	0.017	0.019	0.053	0.118	0.432	0.287	0.000	0.001	0.003	0.118	0.222	0.066	0	0.000	0.000	0.118	0.114	0.020
HRV	0.058	0.098	0.271	1	0.551	0.336	0.007	0.008	0.025	0.092	0.433	0.378	0.000	0.001	0.003	0.092	0.208	0.110	0	0.000	0.001	0.092	0.115	0.028
HUN	0.050	0.092	0.252	1	0.637	0.399	0.008	0.009	0.027	0.177	0.464	0.281	0.000	0.001	0.002	0.177	0.296	0.071	0	0.000	0.000	0.177	0.188	0.018
IDN	0.049	0.086	0.228	1	0.758	0.586	0.013	0.019	0.051	0.124	0.844	0.436	0.000	0.001	0.002	0.124	0.495	0.166	0	0.000	0.000	0.124	0.290	0.066
IND	0.029	0.119	0.304	1	0.538	0.335	0.022	0.028	0.079	0.139	0.530	0.448	0.001	0.002	0.004	0.139	0.158	0.150	0	0.000	0.000	0.139	0.050	0.047
IRL	0.034	0.060	0.161	1	0.391	0.191	0.011	0.009	0.031	0.036	3.049	0.759	0.000	0.001	0.003	0.036	1.193	0.104	0	0.000	0.000	0.036	0.467	0.016
ITA	0.060	0.133	0.347	1	0.435	0.364	0.012	0.012	0.039	0.194	0.239	0.168	0.001	0.002	0.006	0.194	0.069	0.037	0	0.000	0.001	0.194	0.020	0.012
JPN	0.028	0.094	0.247	1	0.464	0.415	0.009	0.007	0.023	0.101	0.298	0.216	0.000	0.001	0.002	0.101	0.101	0.100	0	0.000	0.000	0.101	0.046	0.035
KOR	0.037	0.101	0.264	1	0.409	0.330	0.008	0.010	0.029	0.155	0.174	0.171	0.000	0.001	0.002	0.155	0.062	0.040	0	0.000	0.000	0.155	0.025	0.009
LTU	0.079	0.141	0.390	1	0.498	0.352	0.008	0.005	0.018	0.105	0.523	0.206	0.001	0.001	0.002	0.105	0.261	0.072	0	0.000	0.000	0.105	0.130	0.025
LUX	0.014	0.057	0.146	1	0.391	0.249	0.008	0.012	0.033	0.051	1.941	1.089	0.000	0.000	0.001	0.051	0.760	0.271	0	0.000	0.000	0.051	0.297	0.068
LVA	0.048	0.100	0.269	1	0.482	0.359	0.007	0.006	0.020	0.121	0.341	0.197	0.000	0.001	0.002	0.121	0.094	0.041	0	0.000	0.000	0.121	0.026	0.015
MEX	0.032	0.074	0.195	1	0.696	0.430	0.010	0.009	0.026	0.073	0.532	0.434	0.000	0.001	0.002	0.073	0.370	0.096	0	0.000	0.000	0.073	0.258	0.021
MLT	0.027	0.088	0.228	1	0.605	0.404	0.021	0.019	0.057	0.173	0.295	0.272	0.000	0.001	0.002	0.173	0.178	0.075	0	0.000	0.000	0.173	0.108	0.030
NLD	0.071	0.112	0.306	1	0.528	0.368	0.011	0.013	0.038	0.190	0.252	0.226	0.001	0.001	0.002	0.190	0.070	0.068	0	0.000	0.000	0.190	0.036	0.026
NOR	0.059	0.110	0.291	1	0.435	0.368	0.005	0.004	0.014	0.187	0.123	0.111	0.000	0.001	0.002	0.187	0.054	0.041	0	0.000	0.000	0.187	0.023	0.015
POL	0.066	0.123	0.338	1	0.415	0.364	0.011	0.010	0.031	0.187	0.191	0.154	0.001	0.001	0.003	0.187	0.070	0.054	0	0.000	0.000	0.187	0.025	0.022
PRT	0.073	0.172	0.457	1	0.603	0.467	0.007	0.011	0.031	0.132	0.334	0.251	0.001	0.001	0.004	0.132	0.201	0.082	0	0.000	0.001	0.132	0.121	0.038
ROU	0.027	0.061	0.161	1	0.253	0.201	0.008	0.007	0.023	0.182	0.125	0.118	0.000	0.000	0.001	0.182	0.030	0.009	0	0.000	0.000	0.182	0.008	0.001
RUS	0.000	0.051	0.128	1	0.464	0.285	0.005	0.017	0.043	0.226	0.375	0.171	0.000	0.001	0.002	0.226	0.174	0.047	0	0.000	0.000	0.226	0.081	0.013
SVK	0.063	0.121	0.318	1	0.692	0.516	0.008	0.005	0.018	0.153	0.614	0.543	0.000	0.001	0.002	0.153	0.425	0.280	0	0.000	0.000	0.153	0.294	0.145
SVN	0.085	0.175	0.475	1	0.693	0.434	0.010	0.008	0.026	0.211	0.149	0.123	0.001	0.002	0.005	0.211	0.085	0.031	0	0.000	0.001	0.211	0.059	0.010
SWE	0.052	0.093	0.248	1	0.428	0.391	0.009	0.012	0.036	0.218	0.221	0.149	0.000	0.002	0.004	0.218	0.087	0.064	0	0.000	0.000	0.218	0.034	0.027
TUR	0.028	0.124	0.315	1	0.674	0.507	0.015	0.018	0.054	0.136	0.455	0.381	0.000	0.002	0.005	0.136	0.257	0.112	0	0.000	0.001	0.136	0.173	0.048
TWN	0.038	0.116	0.305	1	0.629	0.402	0.016	0.019	0.054	0.057	1.606	1.212	0.000	0.002	0.004	0.057	0.575	0.206	0	0.000	0.001	0.057	0.206	0.043
USA	0.047	0.139	0.356	1	0.529	0.476	0.007	0.006	0.018	0.167	0.303	0.250	0.000	0.001	0.003	0.167	0.144	0.101	0	0.000	0.000	0.167	0.069	0.053
Mean	0.055	0.114	0.304	1	0.514	0.389	0.010	0.012	0.035	0.158	0.368	0.263	0.000	0.001	0.003	0.158	0.168	0.089	0	0.000	0.000	0.158	0.086	0.036
SD	0.022	0.032	0.087	0	0.107	0.083	0.004	0.005	0.014	0.050	0.532	0.249	0.000	0.001	0.001	0.050	0.224	0.090	0	0.000	0.000	0.050	0.109	0.050

[1] The table reports the median, the interquartile range (IQR), the outlier frontier, the first value ($\lambda_{J,1}, \beta_1, \beta_1\lambda_{J,1}, \beta_1\lambda^2_{J,1}$), the maximum value different from $k=1$, and the maximum subdominant value normalised by the corresponding $k=1$.

Source: Authors' calculations based on WIOD tables, 54 industries.

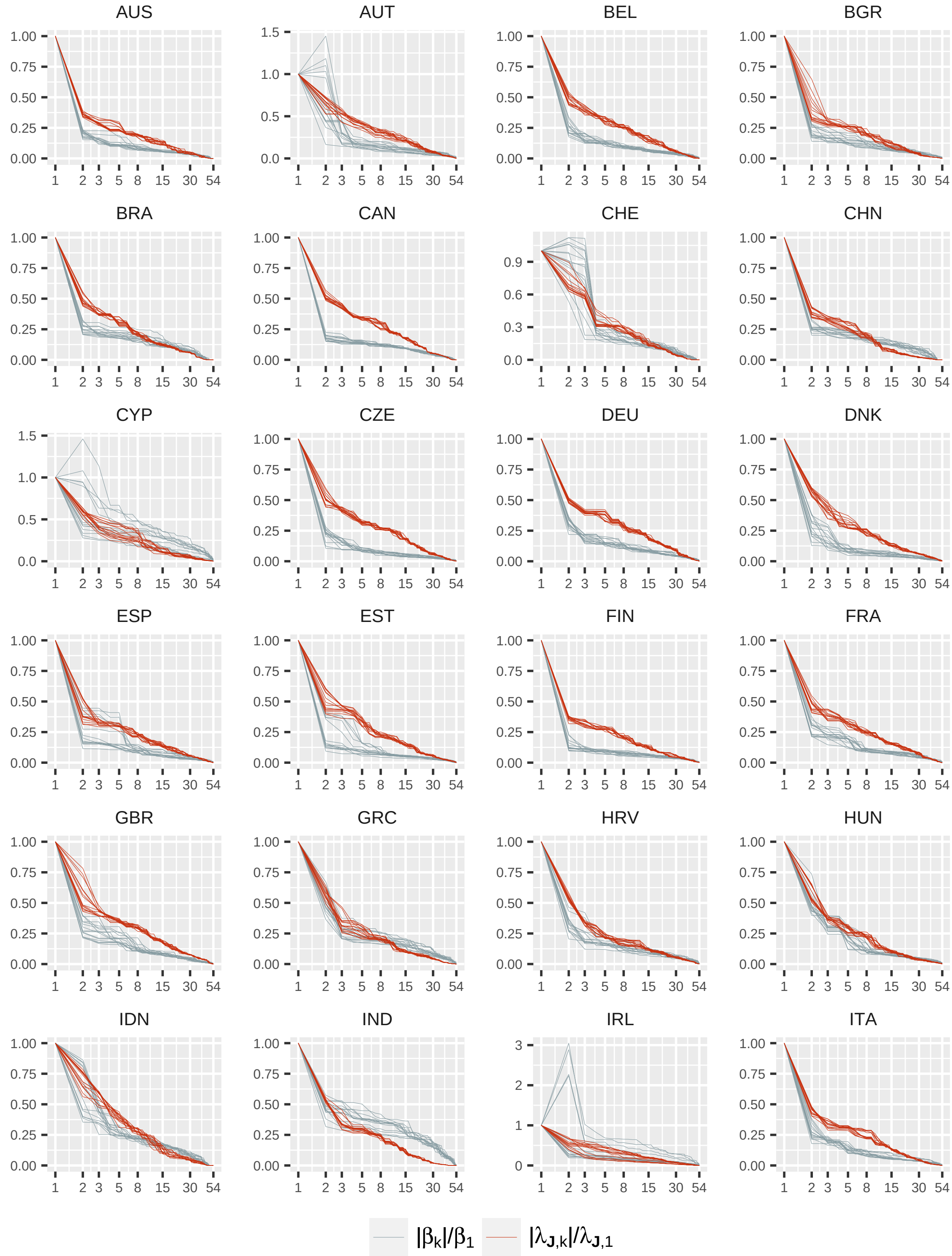


Figure C.10: Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 8 countries, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

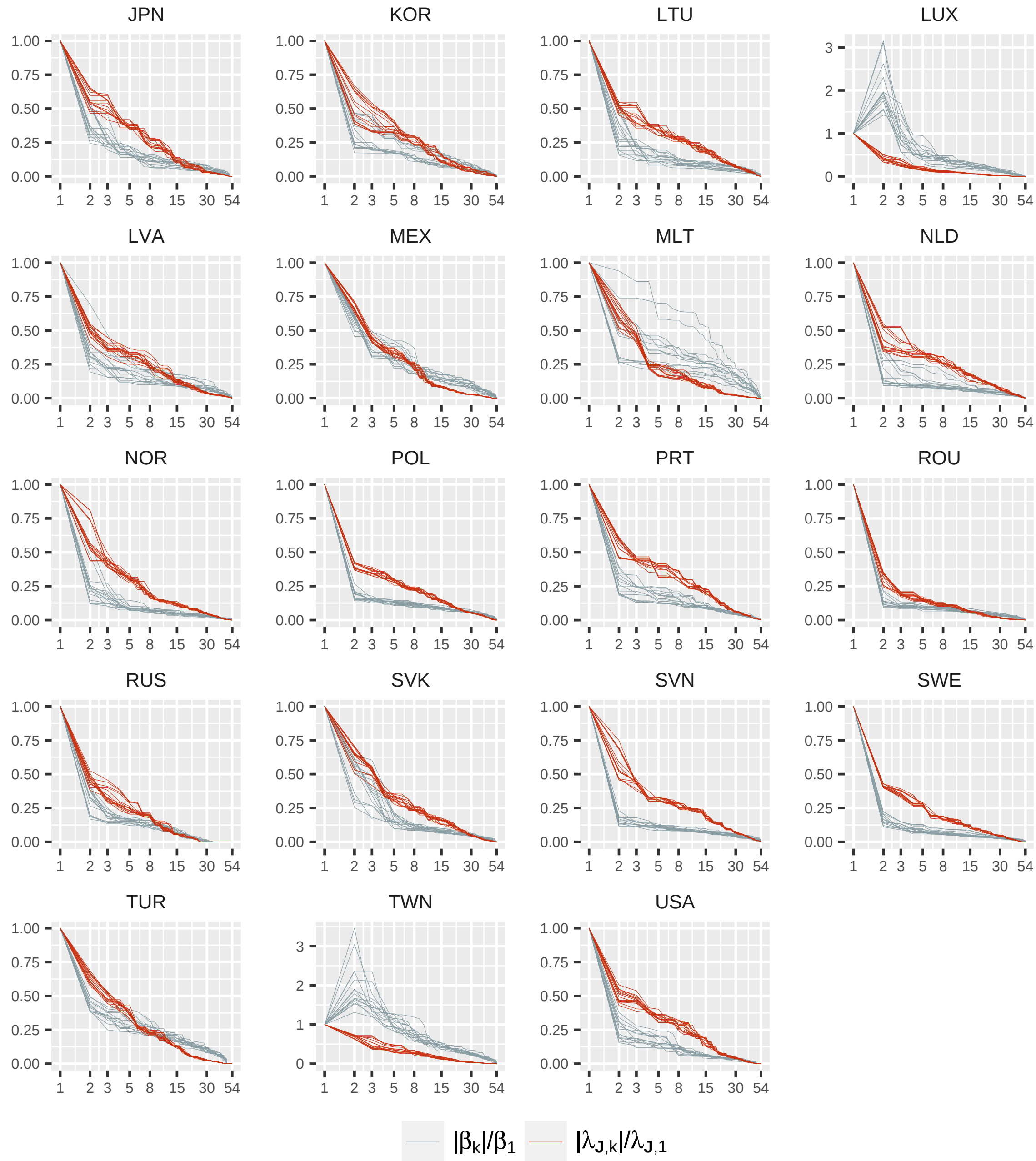


Figure C.11: Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 8 countries, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

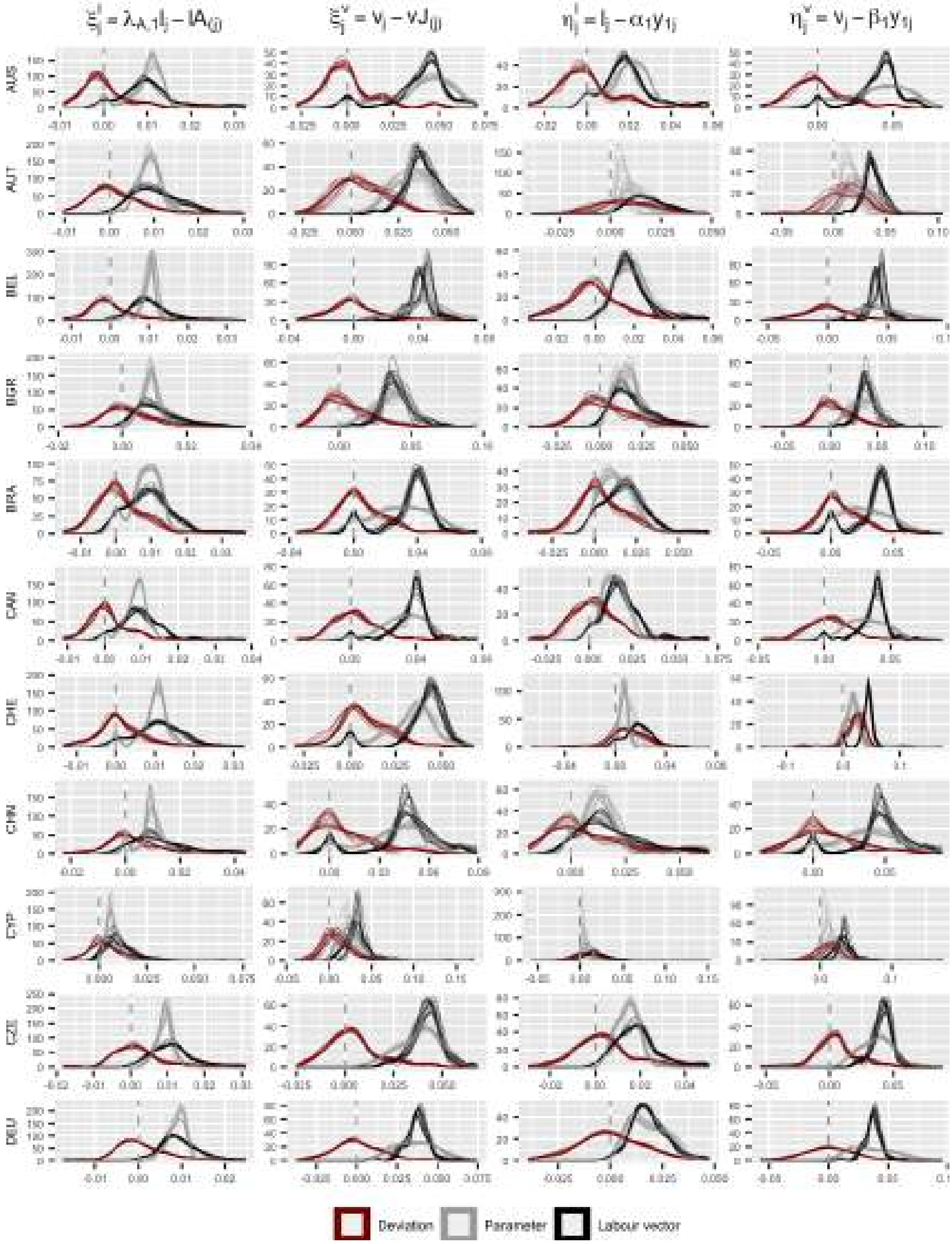


Figure C.12: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi^{\mathbf{l}}_j, \xi^{\mathbf{v}}_j, \eta^{\mathbf{l}}_j, \eta^{\mathbf{v}}_j)$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

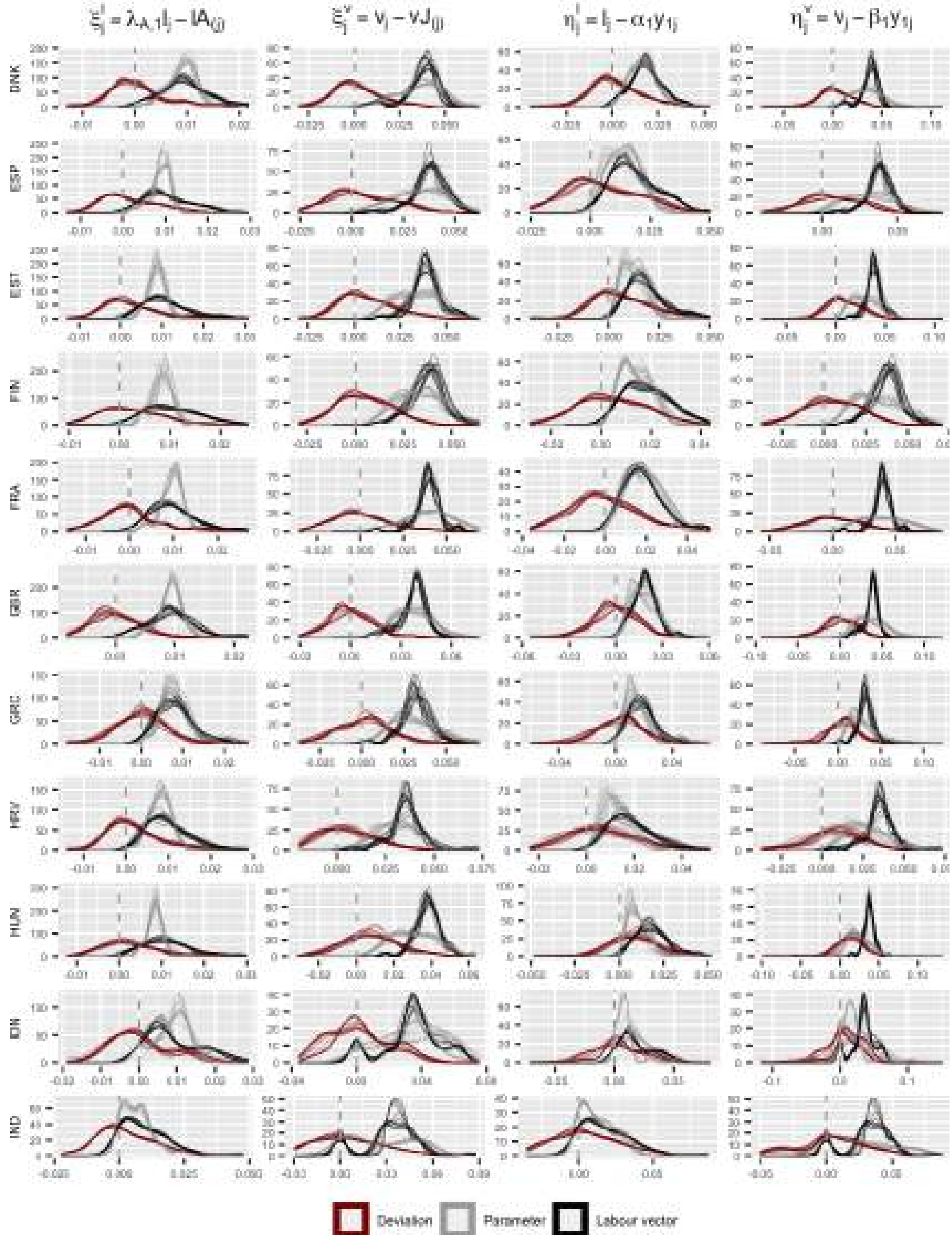


Figure C.13: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi_j^{\mathbf{l}}, \xi_j^{\mathbf{v}}, \eta_j^{\mathbf{l}}, \eta_j^{\mathbf{v}})$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

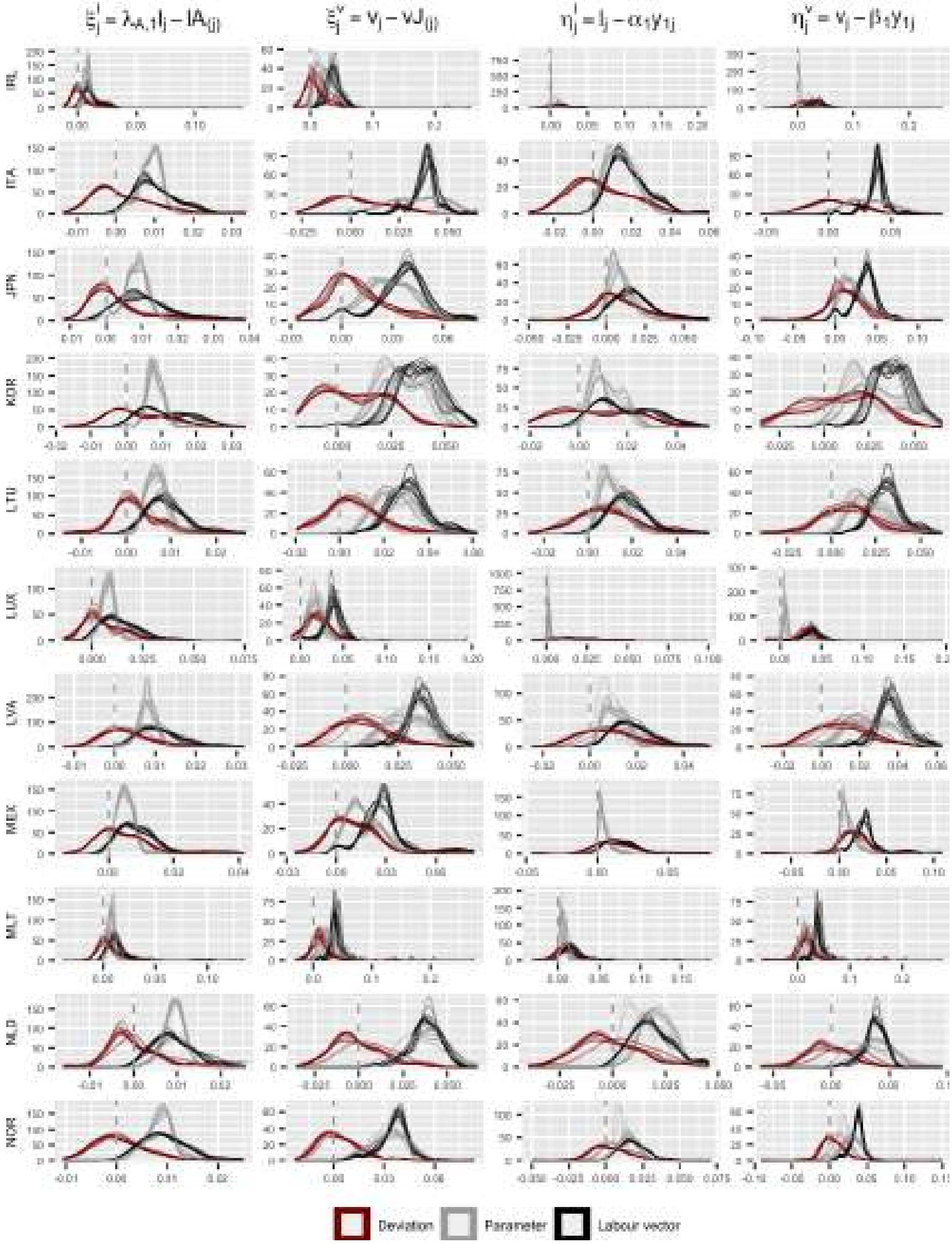


Figure C.14: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi_j^{\mathbf{l}}, \xi_j^{\mathbf{v}}, \eta_j^{\mathbf{l}}, \eta_j^{\mathbf{v}})$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

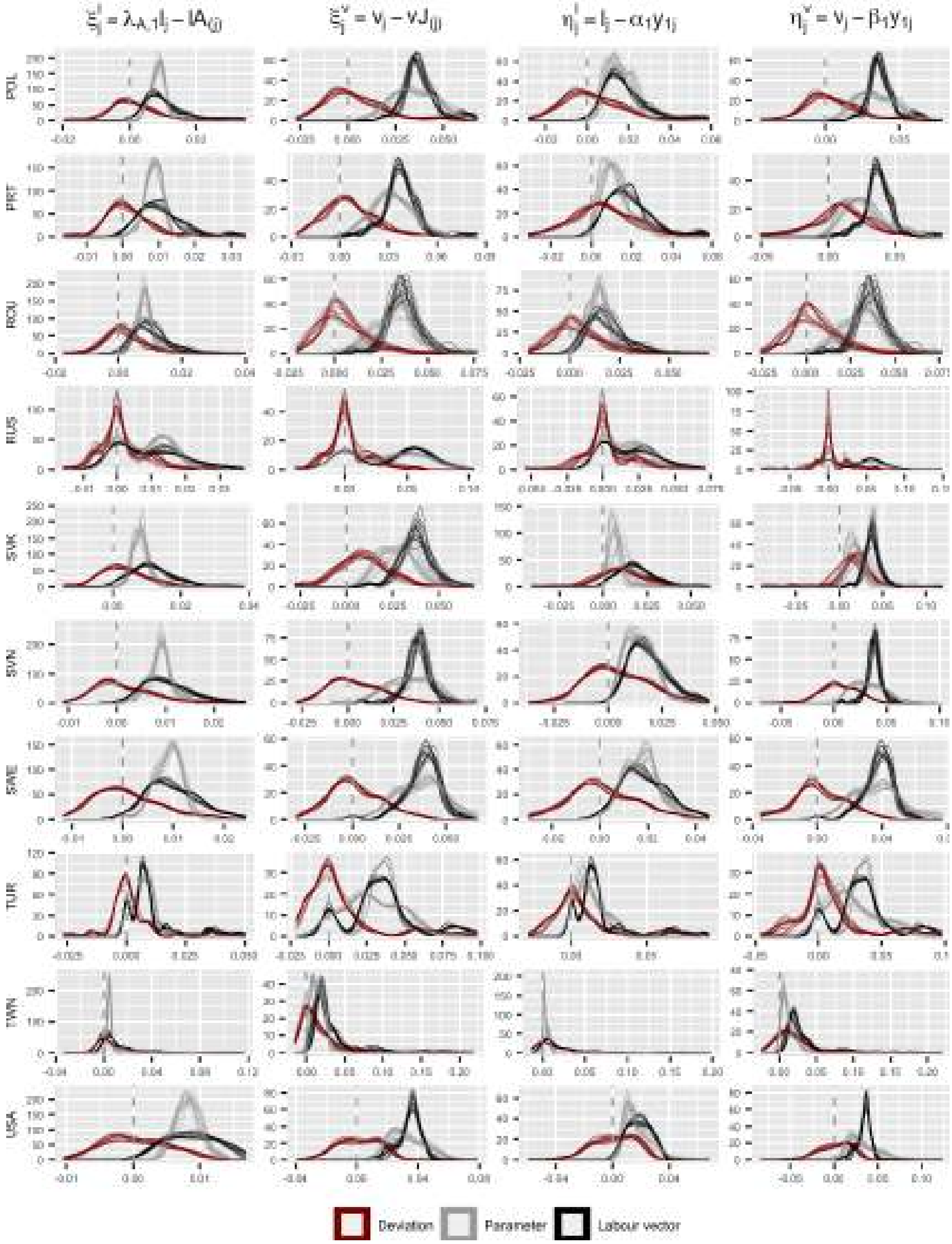


Figure C.15: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi_j^{\mathbf{l}}, \xi_j^{\mathbf{v}}, \eta_j^{\mathbf{l}}, \eta_j^{\mathbf{v}})$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

D Robustness

Sections 2.3 and 3.2 have apprised the relevance of the eigenlabours in conjunction with the eigenvalues to restrict the techniques of production and the productive structures. As stated in Section 3.3, this makes necessary to test the results against alternative labour vectors. To this end, we choose to include two measures of robustness. Section D.1 changes the parameter of interest by focusing "back" onto the direct labour vector $\mathbf{l}$ instead of the vertically integrated one $\mathbf{v}$. That is, the $\alpha_k \equiv \mathbf{l}\mathbf{x}_\mathbf{k}^\mathbf{T}$ eigenlabour. This should clarify the extend to which the input matrix contributes to the highly organized patterns reported in 3.2. Section D.2 addresses the possible bias in the skill-adjusted labour vector by considering a simple metric of the number of people engaged in production (employees + self-employed). In this latter case the parameter of interest will be the β_k.

D.1 The "direct" eigenvalues $\lambda_{\mathbf{A},k}$ and eigenlabours α_k with the skill-adjusted labour vector

Figure D.1 presents the 2D histograms of the empirical distribution of subdominant eigenvalues ($\lambda_{\mathbf{J}k}$) and eigenlabours (β_k) in an analogous way to Figure 2. This time, however, we limit the data display in two ways. First, we consider only the individual countries in the MT2011 sample together with the two aggregations corresponding to all countries in 2011 (WIOD2011) and all countries over all years (WIOD). Second, we concentrate on the direct eigenvalues ($\lambda_{\mathbf{A}k}$) and eigenlabours (α_k). It is possible to observe that the vast majority of the observations are located in the neighbourhood of zero, which seems robust to different cross-sectional and time aggregations. As we get closer to the origin, the clustering becomes more and more populated. This pattern of concentration around zero seems stronger and more symmetric in the case of the eigenlabours than the eigenvalues (*SF3*), which show a higher variability, a larger average magnitude and the presence of an important number of observations with a considerable magnitude (*SF2*). This replicates the conclusions drawn from Figure 2, with the caveat that the degree of variability and the magnitudes of the maxima are larger in the direct than the vertically integrated $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$. This suggests that the statistical tendency towards zero of the subdominant $\beta_k\lambda_{\mathbf{J}k}$ and $\lambda_{\mathbf{J}k}^2\beta_k$ (*SF1*) is strengthened by vertical integration but in no way produced by the interindustry demand structure.

The speed of convergence provides further evidence in favour of *SF2* and *SF3*. It can be evaluated by looking at the distribution of the moduli of the eigenvalues $|\lambda_{\mathbf{A}k}|$ and the eigenlabours $|\alpha_k|$ normalised by their first observations ($\lambda_{\mathbf{A}1}$,α_1), as explained in Section 3.2. Figure D.2 shows the rank plot of the individual countries in the MT2011 sample, where each line represents one monotonically non-increasing country-year-variate; to achieve this, the eigenlabours are rearranged in decreasing

Figure D.1: Complex plane density plot of the distribution of the direct eigenvalues $\lambda_{\mathbf{A}k}$ and eigenlabours α_k 8 countries, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

order. The X-axis is set in logarithmic scale to increase the resolution in the initial steps. For the most part, with the exception of CHN, BRA and, in particular, IND, the $|\alpha_k|$ outrun the $|\lambda_{\mathbf{A}k}|$ in their movement towards zero. For JPN and BRA the two cross at around 0.25 and 0.45, respectively, whereas for CHN and IND we cannot say that the $|\alpha_k|$ outperform the $|\lambda_{\mathbf{A}k}|$ at any point in the distribution, save a couple of fringe cases.

We observe a few small differences with regards to Figure 4. First, in general, the distributions show a slightly lower rate of decline and a higher variability. For instance, the acceleration peaks for $|\beta_k|$ in between 0.15 and 0.25 in AUS, DEU, FRA, JPN, USA, CHN and BRA, whereas for $|\alpha_k|$ it is around 0.35 for AUS, DEU, FRA and BRA and 0.25 for USA; CHN cannot be said to change speed at any point clearly, and IND, in fact, accelerates as it approaches zero. The case is even stronger for $|\lambda_{\mathbf{A}k}|$, which peaks in between 0.5 and 0.6 for AUS, DEU, FRA and IND, with no obvious turning point for JPN, USA, CHN and BRA, as opposed to $|\lambda_{\mathbf{J}k}|$, where AUS, DEU, FRA, JPN, USA, CHN, BRA and IND peaked in between 0.3 and 0.5. Second, we observe several cases in which the eigenvalues and eigenlabours cross at sufficiently high values, which for countries like JPN, CHN or BRA disappear as we

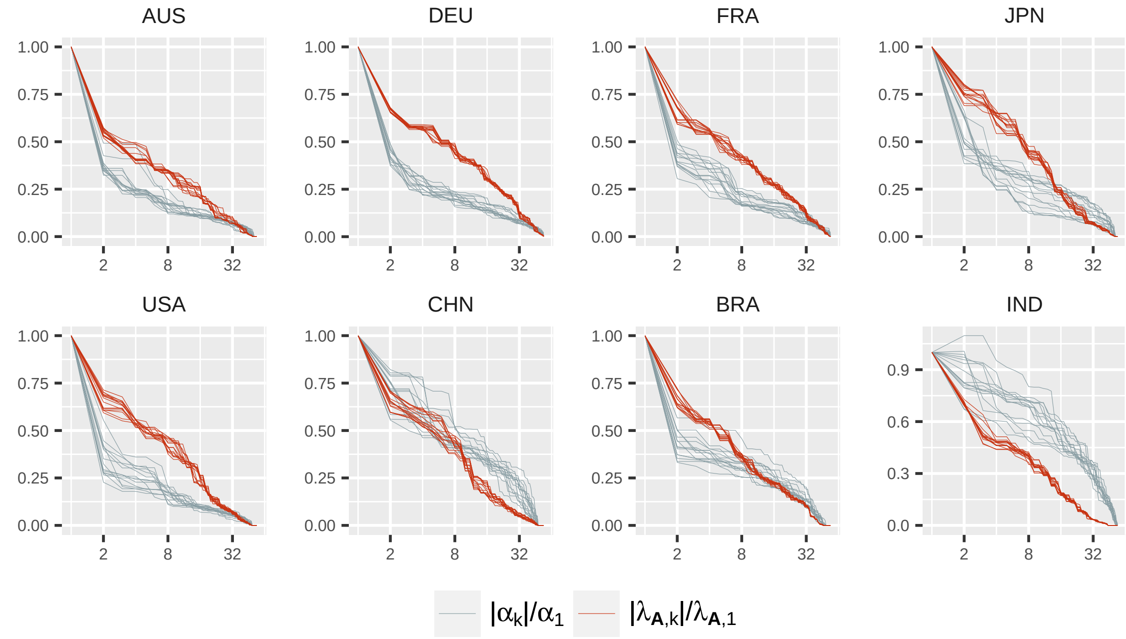


Figure D.2: Rank plot of the moduli of the eigenvalues $|\lambda_{\mathbf{A}k}|/\lambda_{\mathbf{A}1}$ and the eigenlabours $|\alpha_k|/\alpha_1$ for 8 countries, 54 industries. Source: authors' calculations based on the WIOD database, 2016 release.

look at the vertically integrated eigens. Finally, for IND we see that for more than one year there are subdominant values larger than the first, which can be seen as the variate spikes initially to then fall below zero after a few steps.

This evidence seems to posit vertical integration as a relevant force in strengthening the overall patterns found in the $|\alpha_k|$ and $|\lambda_{\mathbf{A}k}|$, but it wrong to uphold this process as the generator of the particular distributions of $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ underlying the near-linearity of price and capital curves.

D.2 Empirical results with persons engaged (EMP) as the labour input

The following data seeks to underscore that the fundamental results regarding the skill-adjusted labour vector hold when we account for the number of people engaged in production (employees + self-employed). Figure D.3 reports the boxplot of the subdominant observations for the individual countries in the MT2011 sample together with the 2D histograms of the complex plane distribution of the pooled MT2011 sample and the aggregation for all countries and years (WIOD). The two aggregations compare nicely to those reported in Figure 2. We can observe that the empirical distribution of the subdominant observations of β_k, $\beta\lambda_{\mathbf{J}k}$ and $\beta\lambda^2_{\mathbf{J}k}$ cluster strongly and increasingly around zero in the way predicted by *SF1* and supported by the previous evidence. Since the $\lambda_{\mathbf{J}k}$ are the same and the variability of the other three parameters is on a level with the labour-adjusted vectors, we can conclude

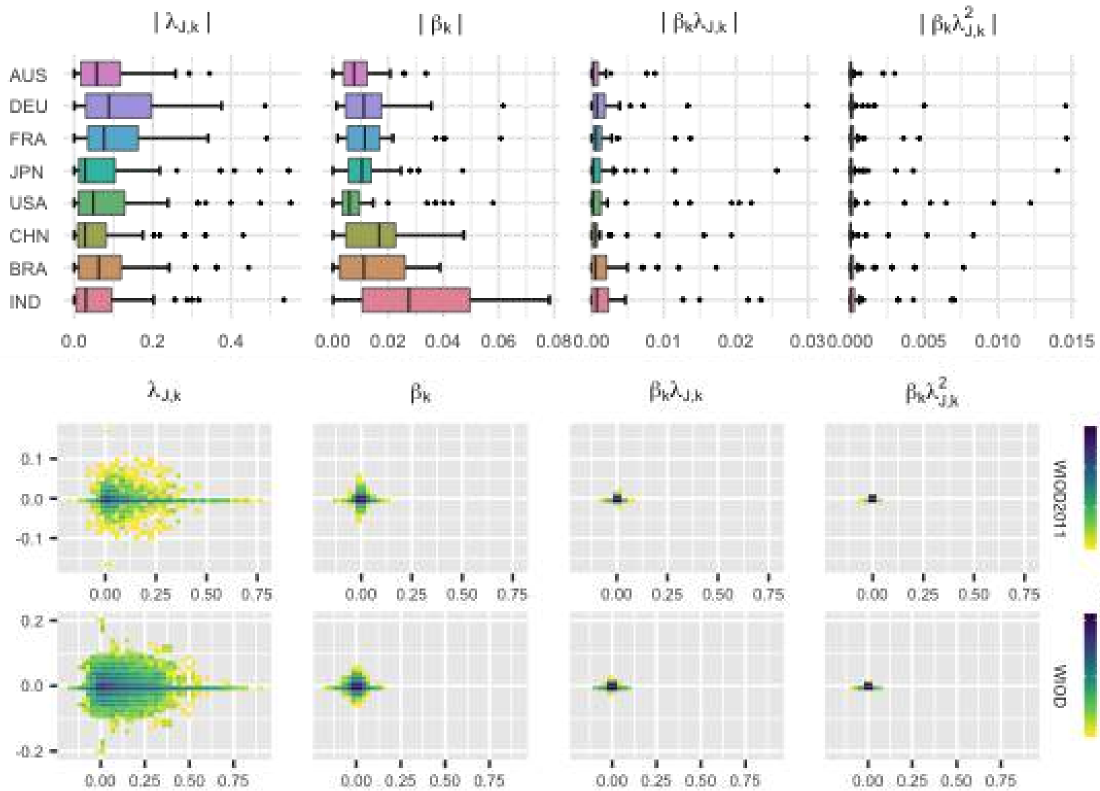


Figure D.3: Boxplot of the empirical distribution of eigenvalues' moduli of $\mathbf{J}$ matrices for the MT2011 and the 2-D histogram of subdominant eigenvalues for the MT2011 and the WIOD aggregations, 54 industries. Source: authors' calculations based on the WIOD database, 2016 release.

again that, in accordance with *SF2*, the $\lambda_{\mathbf{J}k}$ display a larger degree of variability than β_k, contributing less than the latter to the tendency for $\beta\lambda_{\mathbf{J}k} = 0$ and $\beta\lambda^2_{\mathbf{J}k} = 0$. This characterisation applies to every country. The full extent of the sample can be checked in Figures D.5-D.7, whose data does not differ much from Figures C.1-C.3.

From the individual boxplots in Figure D.3 we can evaluate the moduli of the observations outside the neighbourhood around zero for each parameter $|\lambda_{\mathbf{J}k}|, |\beta_k|, |\beta_k\lambda_{\mathbf{J}k}|, |\beta_k\lambda^2_{\mathbf{J}k}|$. Figures D.8-D.13 and Tables D.1-D.3 provide the complete set of results. For $|\beta_k|$, $|\beta_k\lambda_{\mathbf{J}k}|$ and $|\beta_k\lambda^2_{\mathbf{J}k}|$ there is only a small proportion of the observation located outside the vicinity around zero, which can be considered outliers. For the 43 countries in 2011 the average values of the RW (IQR) for $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ are 0.312 (0.086) and 0.046 (0.037); for $|\lambda_{\mathbf{J}k}\beta_k|$ and $|\lambda^2_{\mathbf{J}k}\beta_k|$ are 0.005 (0.003) and 0.0001 (0.0000), respectively. Although their magnitude is small, it is still larger for the vector of persons engaged than for the skill-adjusted vector. In the case of the latter, the maximum value for CHN is close to 0.06, whereas in the former it stays close to 0.10. The same applies to most other countries, such as the USA and, specially, IND, where the maximum value for the persons engaged vector almost doubles that of the skill-adjusted. For the WIOD2011 aggregation, however, Table D.2 show that the average $|\beta_1|$ was 0.143 for the persons engaged and 0.15 for the skill-adjusted vector, with an almost identical standard deviation. Notwithstanding the higher variability that $|\beta_k|$ produces on $|\beta_k \lambda_{\mathbf{J}k}|$ and $|\beta_k \lambda^2_{\mathbf{J}k}|$, it still contrasts sharply with the larger number and magnitude of the subdominant $|\lambda_{\mathbf{J}k}|$.

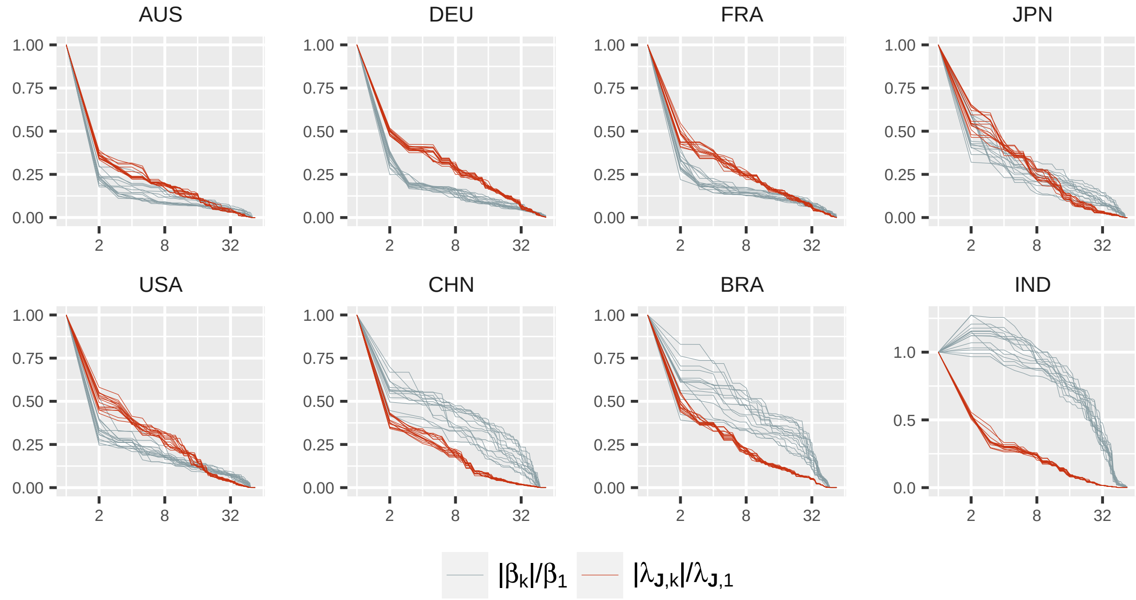


Figure D.4: Rank plot of the moduli of the eigenvalues $|\lambda_{\mathbf{J}k}|/\lambda_{\mathbf{J}1}$ and the eigenlabours $|\beta_k|/\beta_1$ for 8 countries, 54 industries. Source: authors' calculations based on the WIOD database, 2016 release.

We can further asses the speed of convergence by relying again on the rank plot of the individual countries in the MT2011 sample, as reported by Figures D.4 (full results in figures D.14 and D.15); each line represents one country-year-variate arranged in decreasing order. In this case, we can see that the overall patterns reported in Figure 4 hold for the persons engaged vector in AUS, DEU, FRA and USA. In particular, the slope and the inflection point seems to track closely the previously reported $|\lambda_{\mathbf{J}k}|$ and $|\beta_k|$ for these countries. We find two main differences. First, eigenvalues $|\lambda_{\mathbf{J}k}|$ drop faster than eigenlabours $|\beta_k|$ in CHN, BRA and IND, which was not the case for the skill-adjusted vector; other cases are CHE, CYP, HRV, IDN, IRL, LUX, LVA, MEX, TUR and TWN. JPN has some overlapping but overall the two trajectories can still be separated, even if less markedly than before. Second, we find the same phenomena reported in Figure D.2, where the $|\beta_k|$, in that case the $|\alpha_k|$, showed several subdominant values larger than the first; this obtains for AUT, CHE, IDN, IRL, LUX, MEX and TWN. For the MT2011 sample and with persons engaged, the percentage of subdominant $|\lambda_{\mathbf{J}k}|/\lambda_{\mathbf{J}1}$ and $|\beta_k|/\beta_1$ larger than 0.25 is 20.5% and 12.2%, respectively. This informs of a similar though weaker distribution of the β_k, which improves sensibly by correcting differences in skills in the labour force.

Finally, we can consider Figures D.16 to D.19 to evaluate how the deviations from proportionality may vary with a persons engaged labour vector. The red lines correspond to the densities of the deviation coefficients $\xi^{\mathbf{l}}_j$, $\xi^{\mathbf{v}}_j$, $\eta^{\mathbf{l}}_j$, and $\eta^{\mathbf{v}}_j$. The densities with the scaled labour coefficients $\lambda_{\mathbf{A}1} l_j$ and v_j are drawn with black lines.

The grey densities correspond to the quantities of labour contained in the direct $\mathbf{lA}_{(j)}$ and the vertically integrated $\mathbf{vJ}_{(j)}$ means of production and the coefficients of the scaled P-F eigenvector $\alpha_1 y_{1j}$ and $\beta_1 y_{1j}$. There are 15 lines for each colour —one for each country-year-variate. Visual inspection suffices to find that the densities are for the most part smooth and unimodal, with a high degree of symmetry in the IQR (*SF4*) and remarkably time invariant (*SF5*). Furthermore, the scale of the vectors resembles strongly that of Figures C.1 to C.15, and the central values of the deviations are clearly located in a small vicinity around zero (*SF6*). These are all additional pieces of evidence indicating that the results presented in Sections 3 and 4 seem robust, and that the main features of the distribution of subdominant $\lambda_{\mathbf{J}k}$, β_k, $\beta_k\lambda_{\mathbf{J}k}$ and $\beta_k\lambda^2_{\mathbf{J}k}$ are indifferent to the choice of alternative labour vectors.

D.3 Detailed empirical evidence for persons engaged

The following figures and tables expand the selected results presented in the previous section.

D.3.1 Complex plane

Figure D.5. Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for AUS, AUT, BEL, BGR, BRA, CAN, CHE, CHN, CYP, CZE,, DEU, DNK, ESP, and EST.

Figure D.6. Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for FIN, FRA, GBR, GRC, HRV, HUN, IDN, IND, IRL, ITA, JPN, KOR, LTU, and LUX.

Figure D.7. Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for LVA, MEX, MLT, NLD, NOR, POL, PRT, ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

D.3.2 Boxplots

Figure D.8. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for AUS, AUT, BEL, BGR, BRA, CAN, and CHE.

Figure D.9. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for CHN, CYP, CZE, DEU, DNK, ESP, and EST.

Figure D.10. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for IN, FRA, GBR, GRC, HRV, HUN, and IDN.

Figure D.11. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for IND, IRL, ITA, JPN, KOR, LTU, and LUX.

Figure D.12. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for LVA, MEX, MLT, NLD, NOR, POL, and PRT.

Figure D.13. Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, for ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

D.3.3 Descriptive statistics

Table D.1. Descriptive statistics of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ for the year 2000, full WIOD sample.

Table D.2. Descriptive statistics of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ for the year 2011, full WIOD sample.

Table D.3. Descriptive statistics of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$ for the year 2014, full WIOD sample.

D.3.4 Rankplots

Figure D.14. Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 54 industries; 2000–2014, for AUS, AUT, BEL, BGR, BRA, CAN, CHE, CHN, CYP, CZE, DEU, DNK, ESP, EST, FIN, FRA, GBR, GRC, HRV, HUN, IDN, IND, IRL, and ITA.

Figure D.15. Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 54 industries; 2000–2014, for JPN, KOR, LTU, LUX, LVA, MEX, MLT, NLD, NOR, POL, PRT, ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

D.3.5 Deviations from proportionality

Figure D.16. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix **J**, 2000-2014, 54 industries, for AUS, AUT, BEL, BGR, BRA, CAN, CHE, CHN, CYP, CZE, and DEU.

Figure D.17. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix **J**, 2000-2014, 54 industries, for DNK, ESP, EST, FIN, FRA, GBR, GRC, HRV, HUN, IDN, and IND.

Figure D.18. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix **J**, 2000-2014, 54 industries, for IRL, ITA, JPN, KOR, LTU, LUX, LVA, MEX, MLT, and NLD.

Figure D.19. Empirical densities of the deviations from proportionality between the labour vectors and the left-hand Perron-Frobenius eigenvector of matrix **J**, 2000-2014, 54 industries, for POL, PRT, ROU, RUS, SVK, SVN, SWE, TUR, TWN, and USA.

Figure D.5: Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

Figure D.6: Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

Figure D.7: Complex plane density plot of the distribution of eigenvalues $\lambda_{\mathbf{J}k}$, the eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

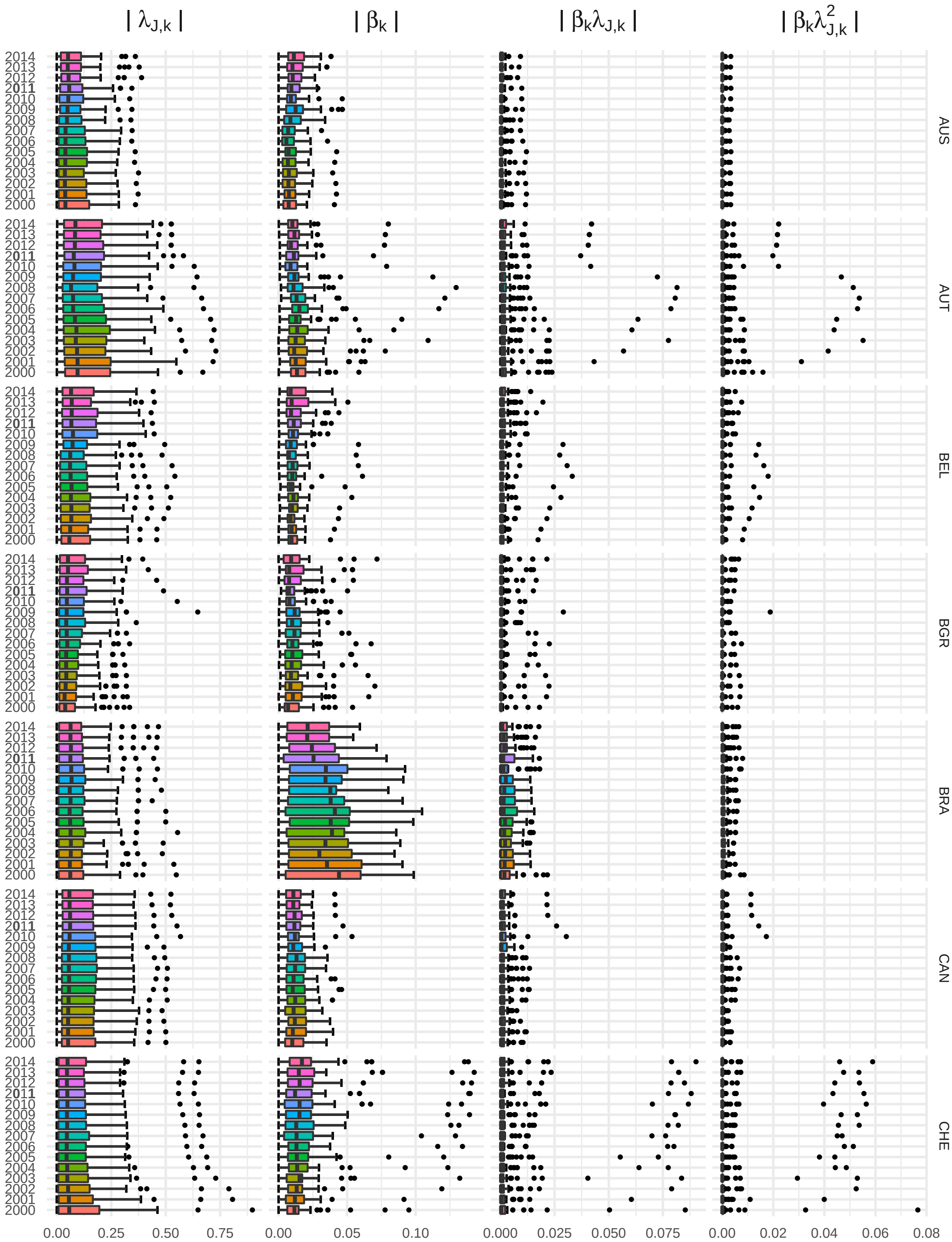


Figure D.8: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda_{\mathbf{J}k}^2\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

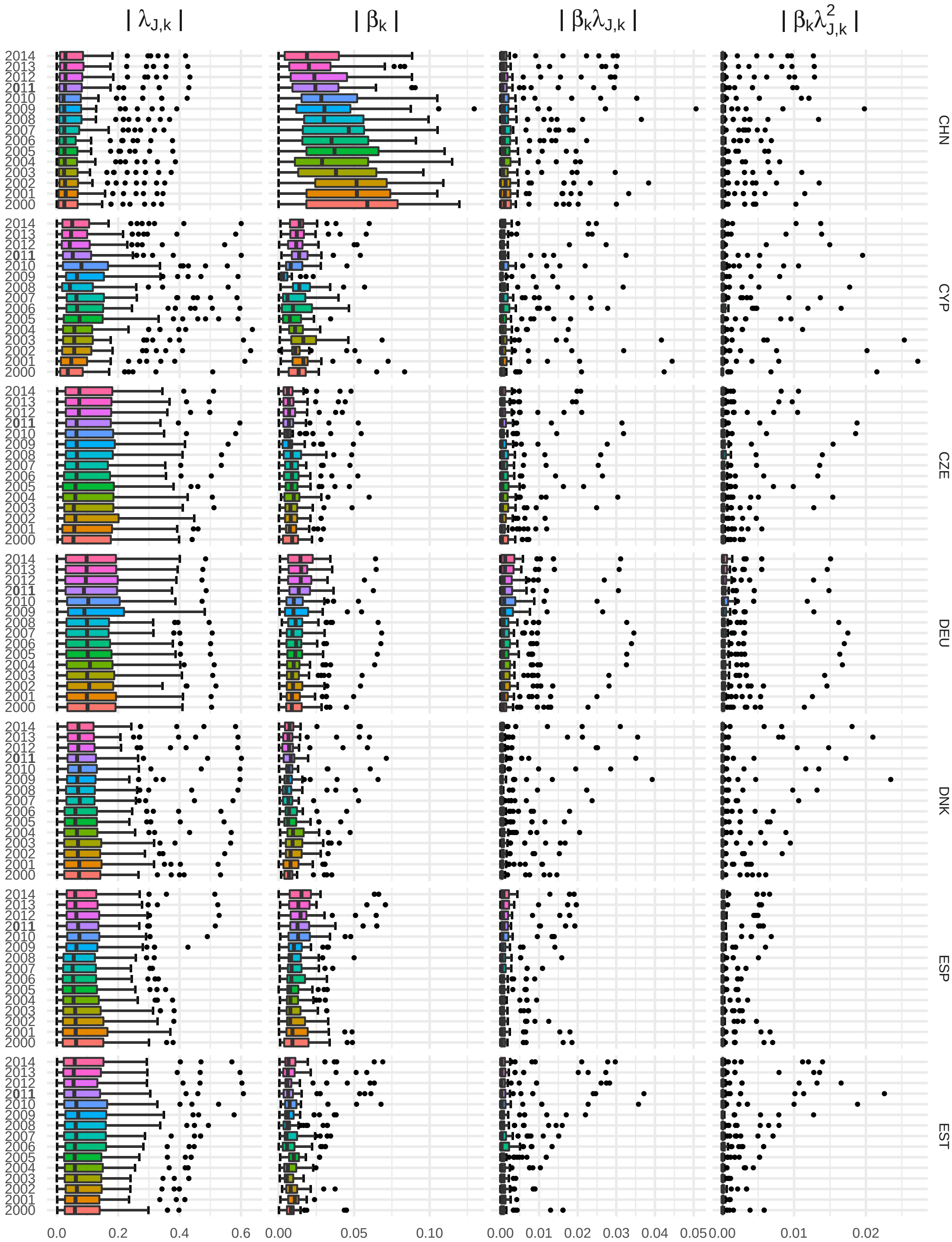


Figure D.9: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

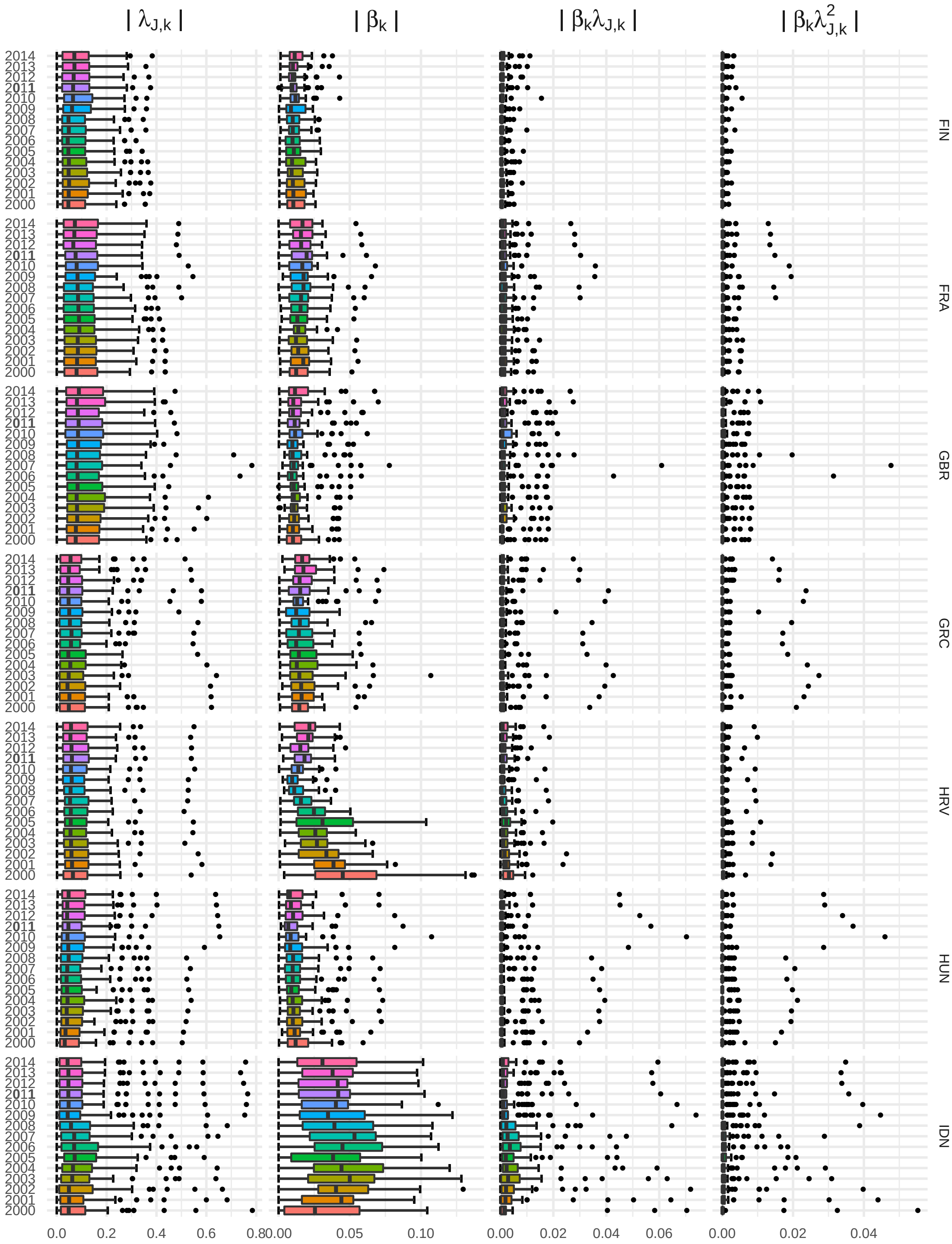


Figure D.10: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

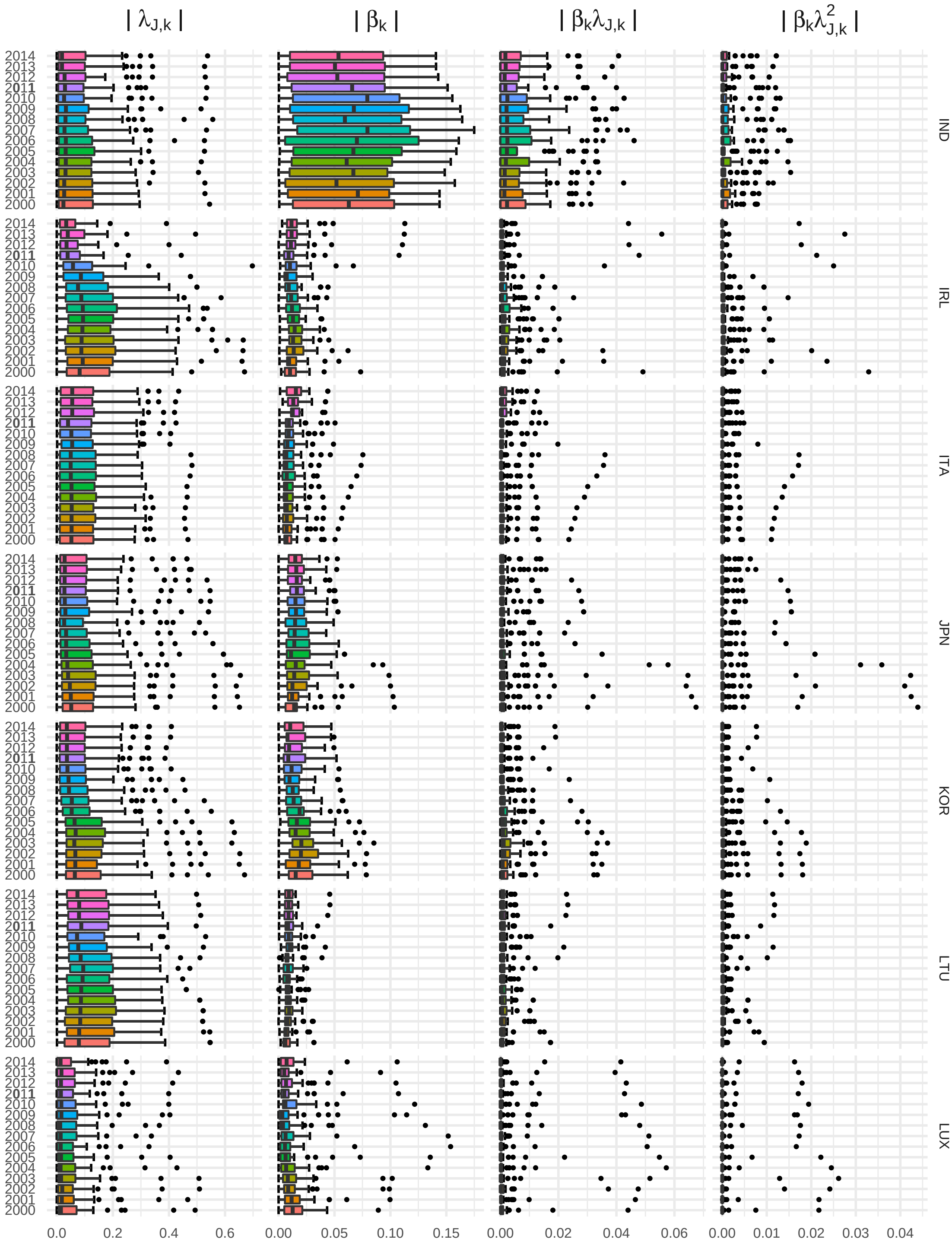


Figure D.11: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

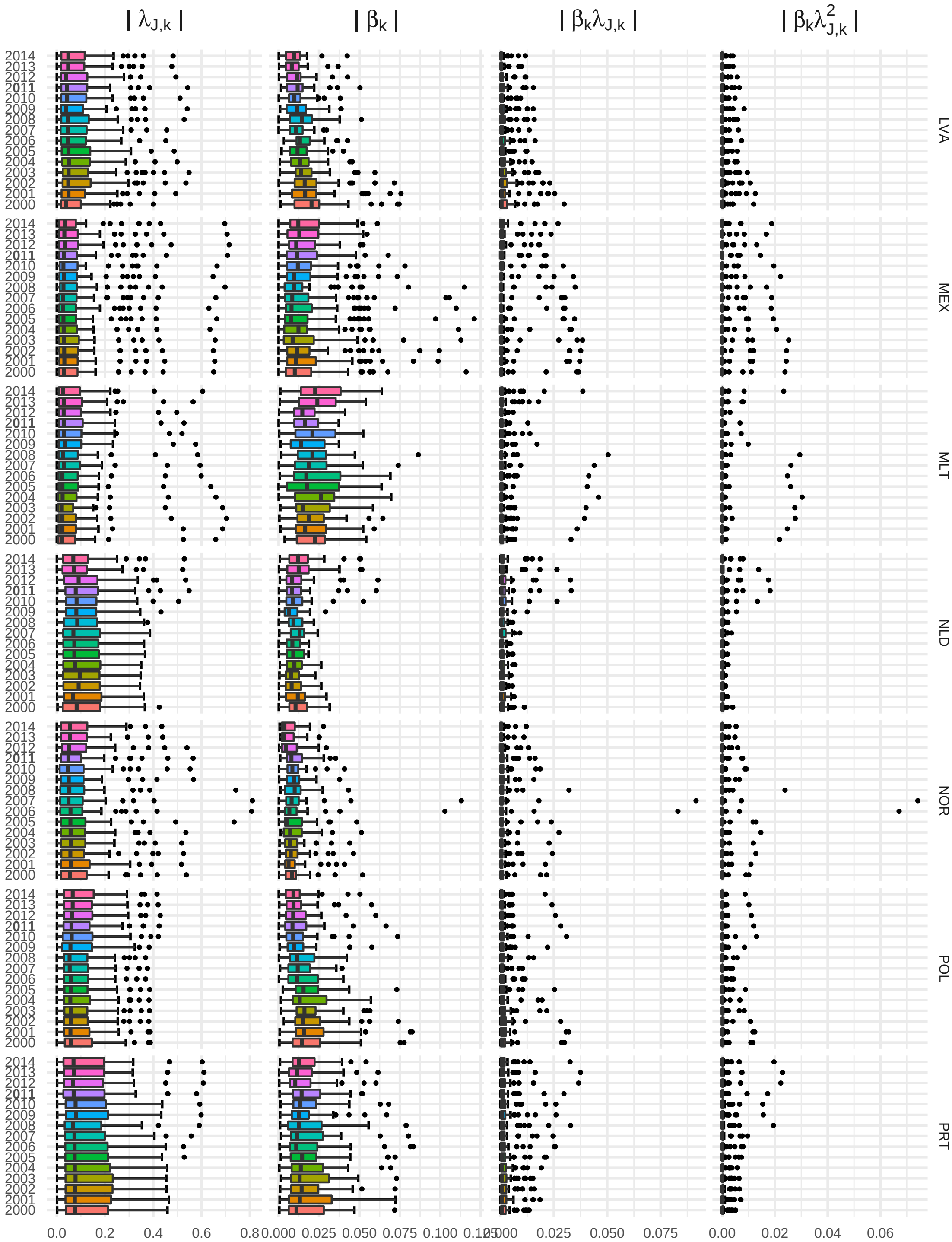


Figure D.12: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

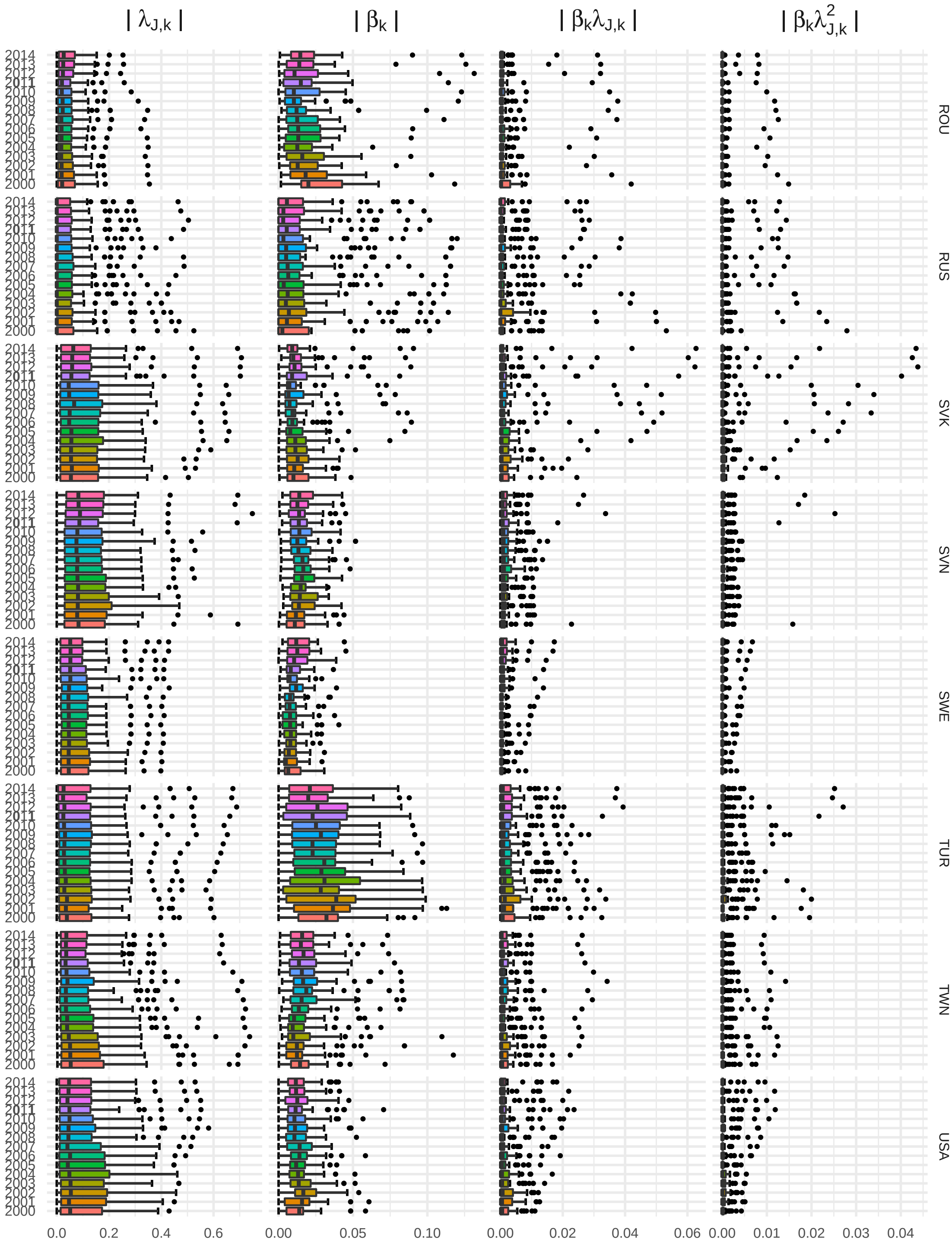


Figure D.13: Boxplot of the empirical moduli of the subdominant eigenvalues $\lambda_{\mathbf{J}k}$, eigenlabours β_k, and their interactions, $\lambda_{\mathbf{J}k}\beta_k$ and $\lambda^2_{\mathbf{J}k}\beta_k$, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

Table D.1: Descriptive statistics for 43 countries, year 2000[1,2]

Country	$\|\lambda_{J,k}\|$						$\|\beta_k\|$						$\|\beta_k\lambda_{J,k}\|$						$\|\beta_k\lambda^2_{J,k}\|$					
	Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$
AUS	0.040	0.140	0.359	1	0.361	0.285	0.007	0.009	0.027	0.187	0.218	0.111	0.000	0.001	0.003	0.187	0.062	0.028	0	0.000	0.000	0.187	0.018	0.010
AUT	0.101	0.217	0.582	1	0.670	0.567	0.014	0.011	0.037	0.162	0.362	0.257	0.001	0.003	0.007	0.162	0.147	0.138	0	0.001	0.002	0.162	0.099	0.074
BEL	0.061	0.142	0.376	1	0.459	0.382	0.010	0.007	0.024	0.194	0.195	0.100	0.000	0.001	0.004	0.194	0.090	0.022	0	0.000	0.001	0.194	0.041	0.007
BGR	0.039	0.084	0.222	1	0.334	0.309	0.007	0.010	0.030	0.151	0.357	0.273	0.000	0.000	0.001	0.151	0.119	0.084	0	0.000	0.000	0.151	0.040	0.026
BRA	0.062	0.111	0.288	1	0.548	0.395	0.044	0.054	0.142	0.140	0.704	0.669	0.002	0.004	0.010	0.140	0.155	0.142	0	0.001	0.001	0.140	0.061	0.052
CAN	0.058	0.157	0.416	1	0.501	0.419	0.010	0.014	0.040	0.213	0.164	0.159	0.001	0.001	0.004	0.213	0.049	0.045	0	0.000	0.001	0.213	0.015	0.014
CHE	0.064	0.187	0.476	1	0.898	0.648	0.012	0.010	0.032	0.144	0.660	0.539	0.000	0.002	0.005	0.144	0.592	0.350	0	0.000	0.001	0.144	0.532	0.227
CHN	0.025	0.062	0.161	1	0.341	0.315	0.059	0.062	0.176	0.211	0.566	0.507	0.001	0.002	0.006	0.211	0.142	0.095	0	0.000	0.000	0.211	0.049	0.022
CYP	0.037	0.075	0.199	1	0.508	0.323	0.013	0.011	0.036	0.050	1.681	1.309	0.000	0.001	0.003	0.050	0.854	0.423	0	0.000	0.000	0.050	0.434	0.137
CZE	0.055	0.170	0.447	1	0.441	0.398	0.009	0.010	0.028	0.202	0.139	0.110	0.000	0.002	0.005	0.202	0.036	0.034	0	0.000	0.001	0.202	0.015	0.012
DEU	0.102	0.171	0.464	1	0.504	0.409	0.009	0.010	0.031	0.181	0.249	0.248	0.001	0.003	0.007	0.181	0.125	0.072	0	0.001	0.001	0.181	0.063	0.030
DNK	0.073	0.124	0.336	1	0.533	0.415	0.007	0.005	0.017	0.191	0.184	0.167	0.000	0.001	0.002	0.191	0.076	0.067	0	0.000	0.000	0.191	0.032	0.027
ESP	0.063	0.146	0.386	1	0.379	0.358	0.009	0.017	0.045	0.198	0.247	0.229	0.000	0.001	0.003	0.198	0.094	0.082	0	0.000	0.000	0.198	0.035	0.029
EST	0.061	0.119	0.323	1	0.398	0.361	0.008	0.004	0.017	0.190	0.236	0.228	0.000	0.001	0.002	0.190	0.085	0.068	0	0.000	0.000	0.190	0.031	0.020
FIN	0.048	0.090	0.249	1	0.355	0.272	0.010	0.012	0.036	0.184	0.141	0.116	0.000	0.001	0.003	0.184	0.028	0.019	0	0.000	0.000	0.184	0.010	0.004
FRA	0.086	0.133	0.364	1	0.435	0.380	0.013	0.014	0.043	0.191	0.270	0.188	0.001	0.002	0.004	0.191	0.066	0.055	0	0.000	0.001	0.191	0.025	0.024
GBR	0.083	0.157	0.432	1	0.483	0.438	0.010	0.010	0.031	0.191	0.223	0.205	0.001	0.001	0.002	0.191	0.090	0.080	0	0.000	0.001	0.191	0.039	0.033
GRC	0.051	0.101	0.264	1	0.621	0.348	0.015	0.011	0.037	0.115	0.471	0.278	0.001	0.001	0.004	0.115	0.292	0.065	0	0.000	0.000	0.115	0.181	0.021
HRV	0.066	0.114	0.312	1	0.539	0.335	0.047	0.043	0.133	0.121	1.138	1.125	0.003	0.004	0.010	0.121	0.101	0.077	0	0.000	0.001	0.121	0.055	0.024
HUN	0.032	0.078	0.208	1	0.503	0.385	0.012	0.015	0.043	0.147	0.404	0.303	0.000	0.001	0.002	0.147	0.203	0.114	0	0.000	0.000	0.147	0.102	0.044
IDN	0.049	0.097	0.260	1	0.785	0.558	0.033	0.054	0.139	0.135	0.773	0.713	0.002	0.002	0.006	0.135	0.521	0.431	0	0.000	0.000	0.135	0.409	0.241
IND	0.029	0.129	0.329	1	0.545	0.296	0.063	0.091	0.239	0.142	1.016	0.963	0.002	0.009	0.022	0.142	0.220	0.199	0	0.001	0.003	0.142	0.056	0.054
IRL	0.082	0.163	0.442	1	0.669	0.481	0.010	0.011	0.032	0.108	0.682	0.379	0.001	0.002	0.004	0.108	0.456	0.182	0	0.000	0.001	0.108	0.306	0.088
ITA	0.052	0.127	0.329	1	0.468	0.345	0.007	0.007	0.023	0.190	0.265	0.214	0.000	0.001	0.003	0.190	0.124	0.070	0	0.000	0.000	0.190	0.058	0.024
JPN	0.056	0.125	0.333	1	0.651	0.564	0.014	0.010	0.032	0.167	0.619	0.320	0.001	0.001	0.003	0.167	0.404	0.180	0	0.000	0.000	0.167	0.263	0.102
KOR	0.064	0.134	0.369	1	0.670	0.540	0.015	0.021	0.062	0.109	0.718	0.566	0.001	0.003	0.007	0.109	0.306	0.295	0	0.000	0.001	0.109	0.165	0.121
LTU	0.079	0.160	0.429	1	0.547	0.386	0.006	0.006	0.019	0.110	0.288	0.153	0.000	0.001	0.002	0.110	0.158	0.037	0	0.000	0.000	0.110	0.086	0.011
LUX	0.017	0.067	0.174	1	0.493	0.417	0.014	0.016	0.046	0.050	1.780	0.866	0.000	0.001	0.002	0.050	0.878	0.361	0	0.000	0.000	0.050	0.433	0.151
LVA	0.039	0.093	0.249	1	0.400	0.304	0.020	0.017	0.052	0.075	0.990	0.977	0.001	0.003	0.008	0.075	0.396	0.232	0	0.000	0.001	0.075	0.159	0.055
MEX	0.030	0.078	0.207	1	0.650	0.441	0.010	0.025	0.067	0.084	1.388	0.809	0.000	0.001	0.002	0.084	0.439	0.428	0	0.000	0.000	0.084	0.285	0.132
MLT	0.022	0.070	0.182	1	0.659	0.525	0.023	0.017	0.054	0.099	0.546	0.505	0.000	0.001	0.003	0.099	0.333	0.068	0	0.000	0.000	0.099	0.219	0.015
NLD	0.082	0.154	0.410	1	0.425	0.365	0.010	0.012	0.036	0.212	0.149	0.121	0.001	0.002	0.005	0.212	0.053	0.030	0	0.000	0.001	0.212	0.019	0.009
NOR	0.060	0.114	0.304	1	0.537	0.416	0.009	0.007	0.021	0.173	0.300	0.202	0.000	0.001	0.003	0.173	0.125	0.108	0	0.000	0.000	0.173	0.058	0.052
POL	0.057	0.112	0.314	1	0.387	0.380	0.015	0.017	0.051	0.192	0.405	0.392	0.001	0.002	0.005	0.192	0.157	0.149	0	0.000	0.001	0.192	0.061	0.057
PRT	0.079	0.193	0.521	1	0.459	0.441	0.011	0.028	0.075	0.184	0.390	0.254	0.001	0.002	0.006	0.184	0.073	0.071	0	0.001	0.001	0.184	0.027	0.027
ROU	0.021	0.065	0.166	1	0.354	0.185	0.021	0.027	0.084	0.184	0.645	0.366	0.000	0.003	0.008	0.184	0.229	0.043	0	0.000	0.001	0.184	0.081	0.008
RUS	0.000	0.064	0.159	1	0.525	0.456	0.003	0.021	0.052	0.175	0.581	0.496	0.000	0.001	0.003	0.175	0.305	0.077	0	0.000	0.000	0.175	0.160	0.024
SVK	0.054	0.144	0.373	1	0.504	0.416	0.010	0.014	0.042	0.156	0.311	0.244	0.000	0.003	0.007	0.156	0.157	0.084	0	0.001	0.001	0.156	0.079	0.029
SVN	0.085	0.182	0.485	1	0.693	0.448	0.011	0.013	0.037	0.221	0.184	0.149	0.001	0.001	0.003	0.221	0.103	0.044	0	0.000	0.001	0.221	0.072	0.012
SWE	0.046	0.105	0.279	1	0.398	0.334	0.007	0.011	0.031	0.207	0.149	0.131	0.000	0.001	0.001	0.207	0.039	0.028	0	0.000	0.000	0.207	0.016	0.009
TUR	0.039	0.128	0.330	1	0.601	0.469	0.033	0.026	0.079	0.143	0.643	0.588	0.001	0.005	0.012	0.143	0.228	0.182	0	0.001	0.002	0.143	0.137	0.081
TWN	0.060	0.162	0.423	1	0.693	0.660	0.014	0.012	0.039	0.062	1.159	0.778	0.001	0.003	0.006	0.062	0.270	0.206	0	0.000	0.001	0.062	0.122	0.108
USA	0.053	0.175	0.453	1	0.430	0.388	0.014	0.011	0.034	0.178	0.329	0.273	0.001	0.002	0.006	0.178	0.062	0.057	0	0.001	0.001	0.178	0.024	0.018
Mean	0.055	0.126	0.335	1	0.520	0.408	0.016	0.019	0.055	0.156	0.433	0.341	0.001	0.002	0.005	0.156	0.175	0.107	0	0.000	0.001	0.156	0.095	0.044
SD	0.023	0.040	0.106	0	0.128	0.098	0.014	0.017	0.046	0.047	0.408	0.305	0.001	0.001	0.004	0.047	0.201	0.117	0	0.000	0.001	0.047	0.131	0.056

[1] The table reports the median, the interquartile range (IQR), the outlier frontier, the first value ($\lambda_{J,1}, \beta_1, \beta_1\lambda_{J,1}, \beta_1\lambda^2_{J,1}$), the maximum value different from $k = 1$, and the maximum subdominant value normalised by the corresponding $k = 1$.
Source: Authors' calculations based on WIOD tables, 54 industries.

Table D.2: Descriptive statistics for 43 countries, year 2011[1,2]

Country	$\lvert\lambda_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda^2_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$
AUS	0.057	0.101	0.270	1	0.345	0.292	0.009	0.009	0.029	0.154	0.186	0.182	0.001	0.001	0.002	0.154	0.064	0.032	0	0.000	0.000	0.154	0.022	0.009
AUT	0.079	0.182	0.490	1	0.581	0.535	0.011	0.009	0.027	0.158	0.439	0.205	0.001	0.002	0.005	0.158	0.235	0.080	0	0.000	0.001	0.158	0.126	0.041
BEL	0.069	0.157	0.419	1	0.439	0.398	0.011	0.011	0.036	0.221	0.174	0.154	0.001	0.002	0.005	0.221	0.053	0.052	0	0.000	0.001	0.221	0.019	0.018
BGR	0.048	0.123	0.322	1	0.490	0.303	0.008	0.006	0.021	0.136	0.369	0.236	0.000	0.001	0.002	0.136	0.112	0.059	0	0.000	0.000	0.136	0.034	0.015
BRA	0.062	0.120	0.312	1	0.445	0.363	0.028	0.041	0.106	0.144	0.548	0.535	0.002	0.006	0.016	0.144	0.125	0.104	0	0.001	0.002	0.144	0.056	0.038
CAN	0.063	0.152	0.409	1	0.552	0.445	0.012	0.010	0.031	0.183	0.257	0.138	0.001	0.002	0.005	0.183	0.142	0.036	0	0.000	0.001	0.183	0.078	0.012
CHE	0.050	0.137	0.352	1	0.630	0.559	0.013	0.019	0.053	0.116	1.205	1.197	0.001	0.001	0.004	0.116	0.759	0.669	0	0.000	0.000	0.116	0.479	0.374
CHN	0.028	0.081	0.213	1	0.430	0.334	0.026	0.031	0.089	0.183	0.496	0.485	0.001	0.001	0.004	0.183	0.161	0.132	0	0.000	0.000	0.183	0.057	0.054
CYP	0.052	0.115	0.309	1	0.601	0.379	0.014	0.011	0.036	0.051	1.067	0.715	0.001	0.001	0.003	0.051	0.641	0.271	0	0.000	0.000	0.051	0.385	0.103
CZE	0.065	0.148	0.400	1	0.596	0.396	0.007	0.006	0.019	0.212	0.249	0.157	0.000	0.001	0.004	0.212	0.148	0.062	0	0.000	0.001	0.212	0.088	0.025
DEU	0.092	0.181	0.481	1	0.487	0.375	0.014	0.014	0.043	0.186	0.338	0.196	0.001	0.003	0.009	0.186	0.164	0.064	0	0.001	0.001	0.186	0.080	0.024
DNK	0.067	0.091	0.265	1	0.601	0.491	0.008	0.007	0.021	0.192	0.372	0.102	0.000	0.001	0.003	0.192	0.183	0.038	0	0.000	0.000	0.192	0.090	0.023
ESP	0.071	0.113	0.316	1	0.516	0.301	0.013	0.012	0.039	0.177	0.367	0.316	0.001	0.002	0.005	0.177	0.109	0.095	0	0.000	0.001	0.177	0.032	0.030
EST	0.057	0.136	0.364	1	0.608	0.460	0.007	0.006	0.019	0.154	0.396	0.370	0.000	0.001	0.003	0.154	0.241	0.160	0	0.000	0.001	0.154	0.146	0.073
FIN	0.070	0.119	0.320	1	0.376	0.304	0.010	0.004	0.020	0.196	0.154	0.139	0.001	0.001	0.003	0.196	0.052	0.032	0	0.000	0.000	0.196	0.020	0.010
FRA	0.081	0.127	0.354	1	0.491	0.341	0.020	0.014	0.045	0.174	0.355	0.260	0.001	0.002	0.005	0.174	0.174	0.057	0	0.000	0.001	0.174	0.085	0.019
GBR	0.088	0.146	0.402	1	0.472	0.394	0.011	0.010	0.030	0.155	0.352	0.328	0.001	0.002	0.006	0.155	0.126	0.115	0	0.000	0.001	0.155	0.049	0.038
GRC	0.049	0.091	0.239	1	0.580	0.467	0.015	0.016	0.046	0.094	0.745	0.601	0.001	0.001	0.002	0.094	0.433	0.090	0	0.000	0.000	0.094	0.251	0.014
HRV	0.061	0.102	0.282	1	0.540	0.355	0.019	0.012	0.041	0.089	0.447	0.442	0.001	0.002	0.006	0.089	0.117	0.073	0	0.000	0.001	0.089	0.063	0.014
HUN	0.047	0.088	0.243	1	0.650	0.356	0.007	0.010	0.030	0.156	0.562	0.257	0.000	0.001	0.003	0.156	0.365	0.061	0	0.000	0.000	0.156	0.237	0.015
IDN	0.046	0.120	0.311	1	0.766	0.590	0.042	0.037	0.107	0.087	1.172	0.942	0.001	0.002	0.005	0.087	0.692	0.295	0	0.000	0.001	0.087	0.409	0.169
IND	0.033	0.110	0.281	1	0.535	0.317	0.066	0.088	0.231	0.125	1.207	1.154	0.002	0.008	0.019	0.125	0.320	0.239	0	0.001	0.002	0.125	0.096	0.082
IRL	0.039	0.076	0.203	1	0.444	0.255	0.009	0.008	0.026	0.042	2.558	0.988	0.000	0.001	0.001	0.042	1.135	0.155	0	0.000	0.000	0.042	0.504	0.039
ITA	0.050	0.130	0.337	1	0.425	0.380	0.009	0.009	0.028	0.184	0.274	0.237	0.000	0.001	0.002	0.184	0.085	0.072	0	0.000	0.000	0.184	0.026	0.022
JPN	0.027	0.094	0.246	1	0.547	0.473	0.017	0.012	0.041	0.114	0.442	0.434	0.000	0.001	0.003	0.114	0.237	0.115	0	0.000	0.000	0.114	0.130	0.040
KOR	0.037	0.107	0.278	1	0.386	0.328	0.011	0.020	0.054	0.141	0.369	0.320	0.000	0.001	0.002	0.141	0.080	0.050	0	0.000	0.000	0.141	0.031	0.009
LTU	0.091	0.142	0.398	1	0.497	0.396	0.009	0.007	0.023	0.104	0.336	0.204	0.001	0.001	0.004	0.104	0.167	0.047	0	0.000	0.001	0.104	0.083	0.014
LUX	0.016	0.067	0.170	1	0.399	0.232	0.005	0.007	0.021	0.047	2.259	1.216	0.000	0.000	0.001	0.047	0.901	0.282	0	0.000	0.000	0.047	0.360	0.065
LVA	0.046	0.111	0.298	1	0.543	0.385	0.012	0.010	0.029	0.110	0.457	0.331	0.000	0.002	0.004	0.110	0.140	0.109	0	0.000	0.001	0.110	0.059	0.043
MEX	0.030	0.074	0.194	1	0.708	0.454	0.013	0.019	0.052	0.062	1.088	0.859	0.000	0.001	0.003	0.062	0.342	0.329	0	0.000	0.000	0.062	0.233	0.108
MLT	0.030	0.105	0.272	1	0.527	0.432	0.017	0.015	0.047	0.131	0.285	0.269	0.000	0.001	0.004	0.131	0.098	0.034	0	0.000	0.000	0.131	0.052	0.007
NLD	0.082	0.136	0.375	1	0.548	0.429	0.008	0.009	0.027	0.176	0.343	0.242	0.001	0.002	0.005	0.176	0.188	0.104	0	0.000	0.001	0.176	0.103	0.044
NOR	0.048	0.100	0.268	1	0.565	0.460	0.008	0.010	0.030	0.181	0.195	0.176	0.000	0.002	0.004	0.181	0.090	0.076	0	0.000	0.000	0.181	0.043	0.041
POL	0.061	0.114	0.314	1	0.424	0.358	0.009	0.013	0.036	0.174	0.383	0.267	0.000	0.001	0.003	0.174	0.162	0.035	0	0.000	0.000	0.174	0.069	0.010
PRT	0.073	0.182	0.484	1	0.579	0.460	0.014	0.017	0.051	0.145	0.358	0.354	0.001	0.002	0.004	0.145	0.205	0.141	0	0.000	0.001	0.145	0.119	0.065
ROU	0.022	0.051	0.135	1	0.258	0.172	0.015	0.019	0.050	0.161	0.709	0.309	0.000	0.001	0.002	0.161	0.183	0.047	0	0.000	0.000	0.161	0.047	0.008
RUS	0.000	0.059	0.147	1	0.485	0.312	0.006	0.015	0.037	0.151	0.631	0.568	0.000	0.001	0.002	0.151	0.179	0.177	0	0.000	0.000	0.151	0.087	0.055
SVK	0.060	0.127	0.336	1	0.703	0.524	0.010	0.013	0.039	0.149	0.546	0.406	0.001	0.002	0.004	0.149	0.384	0.166	0	0.000	0.000	0.149	0.270	0.085
SVN	0.089	0.169	0.457	1	0.690	0.426	0.014	0.013	0.040	0.211	0.194	0.170	0.001	0.002	0.006	0.211	0.087	0.043	0	0.000	0.001	0.211	0.060	0.012
SWE	0.051	0.096	0.255	1	0.410	0.376	0.008	0.010	0.029	0.208	0.177	0.115	0.000	0.001	0.003	0.208	0.067	0.019	0	0.000	0.000	0.208	0.025	0.004
TUR	0.026	0.121	0.309	1	0.662	0.525	0.023	0.043	0.110	0.119	0.741	0.685	0.001	0.004	0.009	0.119	0.274	0.166	0	0.000	0.001	0.119	0.182	0.040
TWN	0.035	0.103	0.274	1	0.638	0.371	0.014	0.018	0.052	0.058	1.335	0.839	0.001	0.002	0.006	0.058	0.465	0.164	0	0.000	0.001	0.058	0.162	0.038
USA	0.049	0.120	0.312	1	0.552	0.475	0.011	0.009	0.030	0.135	0.522	0.327	0.000	0.002	0.004	0.135	0.175	0.159	0	0.000	0.001	0.135	0.088	0.058
Mean	0.053	0.117	0.312	1	0.528	0.395	0.014	0.016	0.046	0.143	0.469	0.353	0.001	0.002	0.005	0.143	0.201	0.104	0	0.000	0.001	0.143	0.103	0.040
SD	0.021	0.031	0.086	0	0.107	0.089	0.010	0.014	0.037	0.047	0.512	0.310	0.000	0.001	0.003	0.047	0.239	0.116	0	0.000	0.000	0.047	0.127	0.061

[1] The table reports the median, the interquartile range (IQR), the outlier frontier, the first value ($\lambda_{J,1}, \beta_1, \beta_1\lambda_{J,1}, \beta_1\lambda^2_{J,1}$), the maximum value different from $k=1$, and the maximum subdominant value normalised by the corresponding $k=1$.
Source: Authors' calculations based on WIOD tables, 54 industries.

Table D.3: Descriptive statistics for 43 countries, year 2014[1,2]

Country	$\lvert\lambda_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$	$\lvert\beta_k\lambda^2_{J,k}\rvert$ Median	IQR	RW	$k=1$	Max $k\geq2/k=1$	Sub $k\geq2/k=1$
AUS	0.052	0.092	0.250	1	0.361	0.316	0.011	0.012	0.036	0.161	0.238	0.192	0.001	0.001	0.003	0.161	0.058	0.024	0	0.000	0.000	0.161	0.021	0.008
AUT	0.086	0.174	0.468	1	0.526	0.477	0.010	0.007	0.024	0.173	0.464	0.166	0.001	0.003	0.007	0.173	0.244	0.066	0	0.001	0.001	0.173	0.128	0.026
BEL	0.066	0.143	0.385	1	0.443	0.366	0.009	0.013	0.040	0.222	0.177	0.172	0.001	0.002	0.005	0.222	0.063	0.036	0	0.000	0.001	0.222	0.023	0.012
BGR	0.052	0.127	0.332	1	0.395	0.331	0.009	0.012	0.033	0.135	0.531	0.408	0.000	0.001	0.002	0.135	0.159	0.111	0	0.000	0.000	0.135	0.047	0.037
BRA	0.063	0.120	0.312	1	0.467	0.416	0.022	0.033	0.088	0.140	0.424	0.405	0.001	0.003	0.007	0.140	0.127	0.099	0	0.000	0.001	0.140	0.046	0.038
CAN	0.062	0.153	0.410	1	0.524	0.432	0.011	0.011	0.032	0.187	0.219	0.134	0.001	0.002	0.005	0.187	0.115	0.030	0	0.000	0.001	0.187	0.060	0.010
CHE	0.051	0.149	0.382	1	0.652	0.582	0.017	0.017	0.050	0.133	1.040	1.019	0.001	0.002	0.005	0.133	0.678	0.593	0	0.000	0.001	0.133	0.443	0.345
CHN	0.029	0.079	0.209	1	0.420	0.358	0.021	0.037	0.096	0.208	0.424	0.375	0.001	0.002	0.005	0.208	0.145	0.140	0	0.000	0.001	0.208	0.061	0.046
CYP	0.050	0.097	0.262	1	0.601	0.414	0.014	0.009	0.030	0.047	1.271	0.805	0.001	0.001	0.003	0.047	0.526	0.483	0	0.000	0.000	0.047	0.290	0.218
CZE	0.075	0.153	0.411	1	0.511	0.414	0.006	0.006	0.018	0.216	0.222	0.188	0.000	0.001	0.003	0.216	0.096	0.092	0	0.000	0.001	0.216	0.049	0.038
DEU	0.102	0.162	0.436	1	0.485	0.401	0.014	0.017	0.050	0.180	0.357	0.191	0.001	0.004	0.011	0.180	0.173	0.076	0	0.001	0.002	0.180	0.084	0.031
DNK	0.072	0.102	0.290	1	0.582	0.479	0.007	0.006	0.018	0.192	0.281	0.277	0.000	0.001	0.002	0.192	0.161	0.110	0	0.000	0.000	0.192	0.094	0.043
ESP	0.064	0.103	0.292	1	0.515	0.357	0.015	0.014	0.042	0.157	0.421	0.403	0.001	0.002	0.005	0.157	0.121	0.114	0	0.000	0.001	0.157	0.042	0.036
EST	0.058	0.129	0.346	1	0.570	0.470	0.007	0.007	0.023	0.158	0.437	0.401	0.000	0.001	0.003	0.158	0.188	0.176	0	0.000	0.000	0.158	0.088	0.076
FIN	0.073	0.109	0.297	1	0.383	0.295	0.012	0.010	0.032	0.207	0.183	0.153	0.001	0.001	0.004	0.207	0.054	0.040	0	0.000	0.000	0.207	0.016	0.015
FRA	0.080	0.135	0.368	1	0.489	0.360	0.018	0.015	0.047	0.171	0.318	0.180	0.001	0.002	0.005	0.171	0.155	0.062	0	0.000	0.001	0.171	0.076	0.022
GBR	0.093	0.151	0.417	1	0.475	0.391	0.012	0.013	0.040	0.175	0.385	0.269	0.001	0.002	0.006	0.175	0.150	0.088	0	0.000	0.001	0.175	0.059	0.042
GRC	0.059	0.085	0.226	1	0.515	0.350	0.017	0.011	0.039	0.100	0.534	0.433	0.001	0.001	0.003	0.100	0.275	0.099	0	0.000	0.000	0.100	0.141	0.028
HRV	0.058	0.098	0.271	1	0.551	0.336	0.022	0.015	0.048	0.077	0.560	0.501	0.001	0.003	0.007	0.077	0.214	0.113	0	0.000	0.001	0.077	0.118	0.029
HUN	0.050	0.092	0.252	1	0.637	0.399	0.008	0.013	0.038	0.161	0.438	0.277	0.001	0.001	0.002	0.161	0.279	0.070	0	0.000	0.000	0.161	0.178	0.018
IDN	0.049	0.086	0.228	1	0.758	0.586	0.031	0.042	0.117	0.091	1.118	0.912	0.001	0.003	0.007	0.091	0.655	0.250	0	0.000	0.001	0.091	0.384	0.099
IND	0.029	0.119	0.304	1	0.538	0.335	0.055	0.082	0.216	0.125	1.121	1.094	0.002	0.007	0.018	0.125	0.325	0.216	0	0.001	0.003	0.125	0.097	0.062
IRL	0.034	0.060	0.161	1	0.391	0.191	0.012	0.009	0.030	0.034	3.292	1.424	0.000	0.001	0.001	0.034	1.288	0.147	0	0.000	0.000	0.034	0.504	0.021
ITA	0.060	0.133	0.347	1	0.435	0.364	0.016	0.012	0.039	0.186	0.237	0.148	0.001	0.002	0.005	0.186	0.068	0.048	0	0.000	0.001	0.186	0.020	0.016
JPN	0.028	0.094	0.247	1	0.464	0.415	0.015	0.015	0.046	0.089	0.590	0.486	0.000	0.001	0.003	0.089	0.157	0.152	0	0.000	0.000	0.089	0.071	0.047
KOR	0.037	0.101	0.264	1	0.409	0.330	0.010	0.017	0.047	0.145	0.325	0.318	0.000	0.001	0.003	0.145	0.130	0.042	0	0.000	0.000	0.145	0.053	0.010
LTU	0.079	0.141	0.390	1	0.498	0.352	0.009	0.007	0.023	0.097	0.471	0.157	0.001	0.001	0.003	0.097	0.235	0.051	0	0.000	0.001	0.097	0.117	0.018
LUX	0.014	0.057	0.146	1	0.391	0.249	0.007	0.012	0.033	0.047	2.280	1.318	0.000	0.000	0.001	0.047	0.893	0.328	0	0.000	0.000	0.047	0.349	0.082
LVA	0.048	0.100	0.269	1	0.482	0.359	0.010	0.009	0.027	0.118	0.360	0.227	0.000	0.001	0.002	0.118	0.099	0.072	0	0.000	0.000	0.118	0.035	0.027
MEX	0.032	0.074	0.195	1	0.696	0.430	0.013	0.024	0.066	0.068	0.894	0.763	0.000	0.002	0.004	0.068	0.395	0.305	0	0.000	0.000	0.068	0.275	0.104
MLT	0.027	0.088	0.228	1	0.605	0.404	0.023	0.024	0.074	0.172	0.370	0.359	0.000	0.002	0.004	0.172	0.224	0.119	0	0.000	0.000	0.172	0.135	0.048
NLD	0.071	0.112	0.306	1	0.528	0.368	0.012	0.013	0.038	0.194	0.261	0.259	0.001	0.002	0.004	0.194	0.096	0.077	0	0.000	0.001	0.194	0.041	0.035
NOR	0.059	0.110	0.291	1	0.435	0.368	0.004	0.008	0.022	0.187	0.149	0.104	0.000	0.001	0.002	0.187	0.065	0.038	0	0.000	0.000	0.187	0.028	0.014
POL	0.066	0.123	0.338	1	0.415	0.364	0.009	0.009	0.027	0.170	0.297	0.253	0.001	0.001	0.003	0.170	0.123	0.033	0	0.000	0.000	0.170	0.051	0.009
PRT	0.073	0.172	0.457	1	0.603	0.467	0.013	0.015	0.045	0.130	0.417	0.345	0.001	0.002	0.005	0.130	0.252	0.107	0	0.000	0.001	0.130	0.152	0.050
ROU	0.027	0.061	0.161	1	0.253	0.201	0.015	0.015	0.047	0.177	0.697	0.510	0.000	0.001	0.002	0.177	0.177	0.103	0	0.000	0.000	0.177	0.045	0.021
RUS	0.000	0.051	0.128	1	0.464	0.285	0.006	0.018	0.046	0.170	0.525	0.523	0.000	0.002	0.004	0.170	0.162	0.150	0	0.000	0.000	0.170	0.075	0.043
SVK	0.063	0.121	0.318	1	0.692	0.516	0.009	0.007	0.024	0.147	0.616	0.555	0.000	0.001	0.002	0.147	0.426	0.287	0	0.000	0.000	0.147	0.295	0.148
SVN	0.085	0.175	0.475	1	0.693	0.434	0.015	0.016	0.048	0.210	0.203	0.191	0.001	0.003	0.007	0.210	0.127	0.047	0	0.000	0.001	0.210	0.088	0.013
SWE	0.052	0.093	0.248	1	0.428	0.391	0.012	0.016	0.046	0.214	0.206	0.123	0.001	0.002	0.005	0.214	0.081	0.047	0	0.000	0.000	0.214	0.032	0.020
TUR	0.028	0.124	0.315	1	0.674	0.507	0.023	0.034	0.092	0.118	0.683	0.588	0.001	0.004	0.009	0.118	0.318	0.158	0	0.000	0.001	0.118	0.214	0.041
TWN	0.038	0.116	0.305	1	0.629	0.402	0.016	0.015	0.047	0.063	1.170	0.745	0.000	0.002	0.005	0.063	0.419	0.155	0	0.000	0.001	0.063	0.150	0.032
USA	0.047	0.139	0.356	1	0.529	0.476	0.012	0.012	0.036	0.148	0.271	0.261	0.001	0.002	0.004	0.148	0.122	0.114	0	0.000	0.001	0.148	0.065	0.054
Mean	0.055	0.114	0.304	1	0.514	0.389	0.014	0.016	0.048	0.147	0.449	0.352	0.001	0.002	0.005	0.147	0.193	0.111	0	0.000	0.001	0.147	0.097	0.043
SD	0.022	0.032	0.087	0	0.107	0.083	0.008	0.013	0.034	0.050	0.580	0.323	0.000	0.001	0.003	0.050	0.243	0.118	0	0.000	0.000	0.050	0.120	0.061

[1] The table reports the median, the interquartile range (IQR), the outlier frontier, the first value ($\lambda_{J,1}, \beta_1, \beta_1\lambda_{J,1}, \beta_1\lambda^2_{J,1}$), the maximum value different from $k=1$, and the maximum subdominant value normalised by the corresponding $k=1$.
Source: Authors' calculations based on WIOD tables, 54 industries.

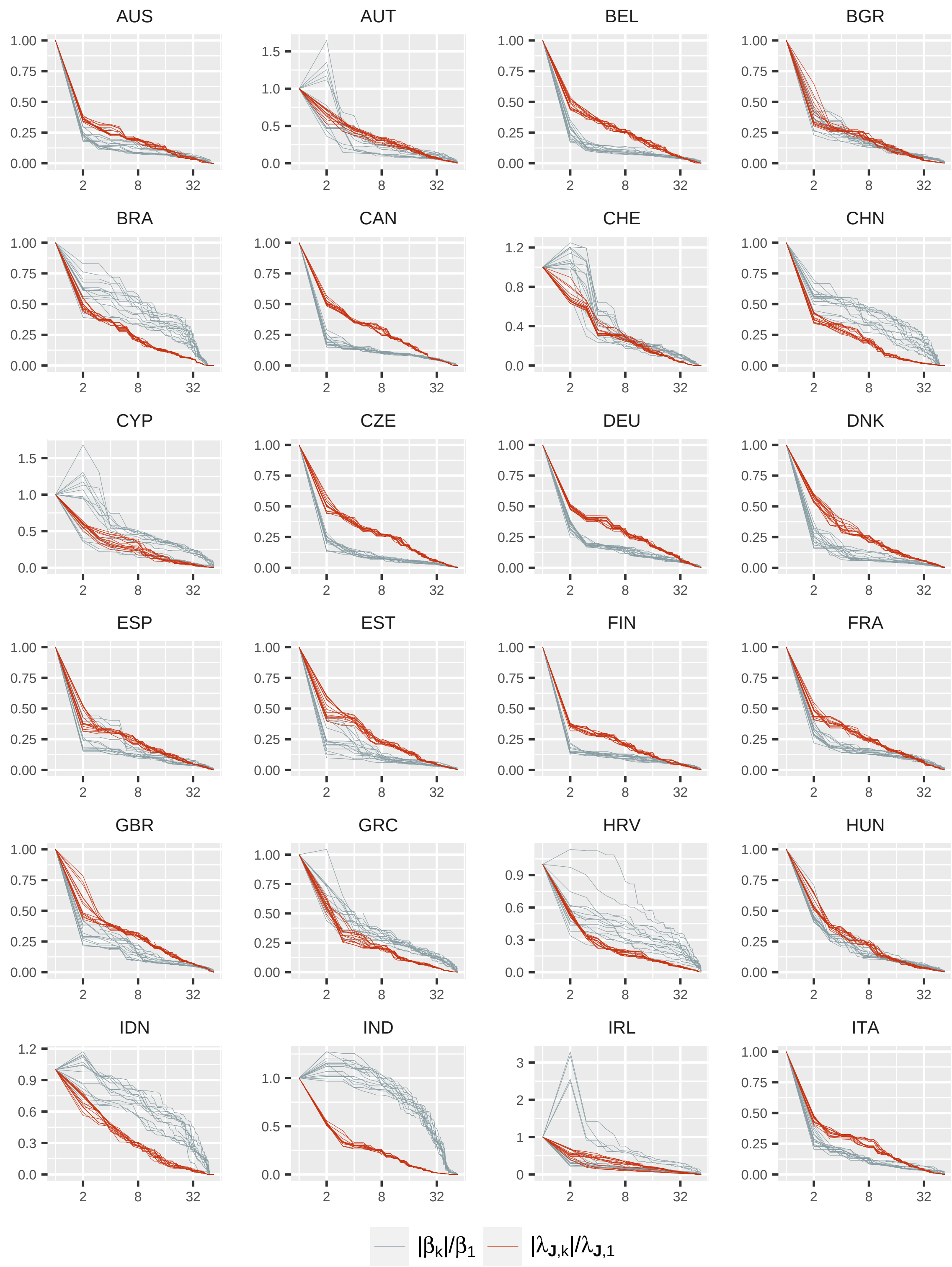


Figure D.14: Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 28 countries, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

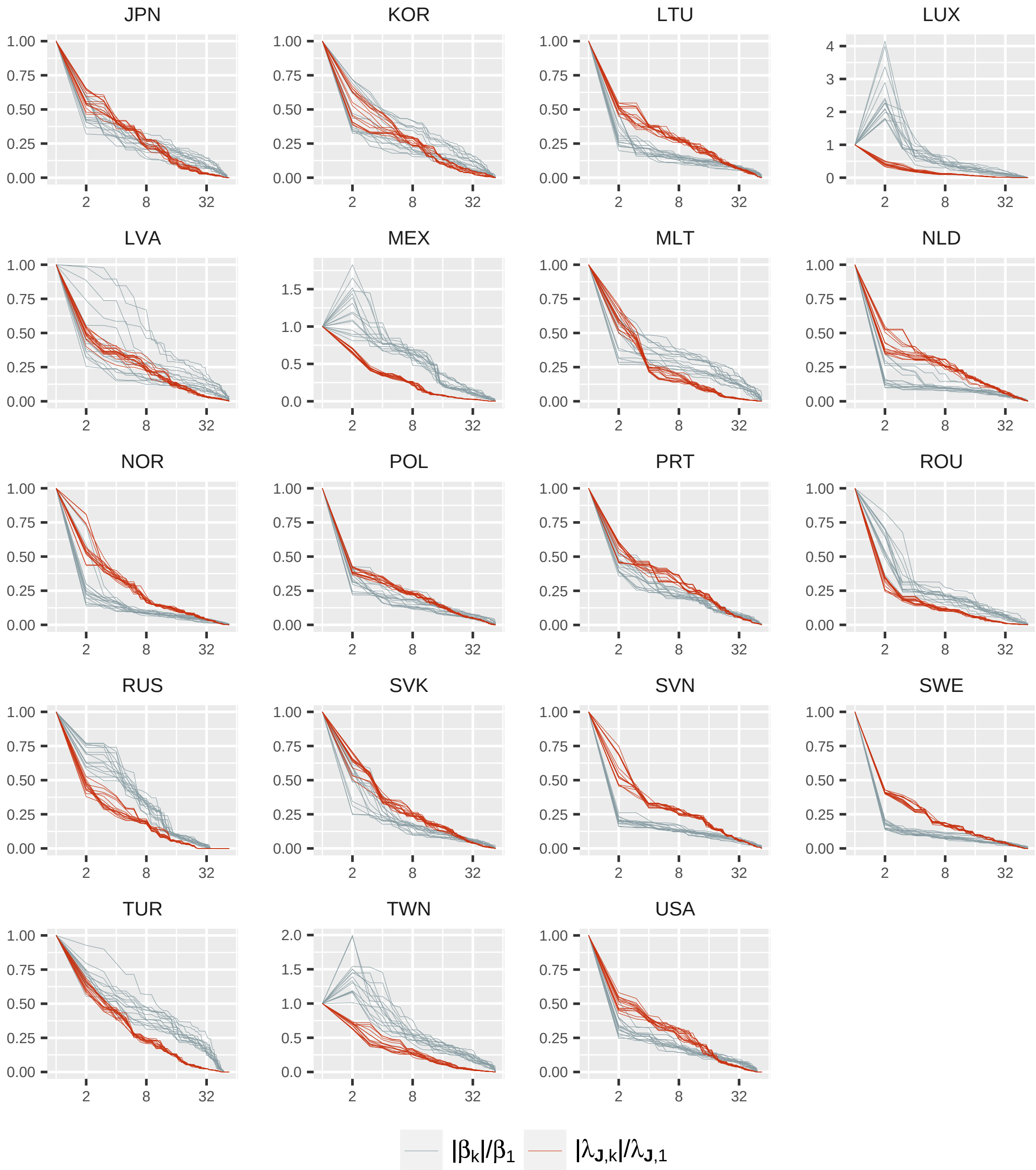


Figure D.15: Rank plot of the empirical distribution of the moduli of the eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenlabours β_k, 15 countries, 54 industries, 2000–2014. Source: authors' calculations based on the WIOD database, 2016 release.

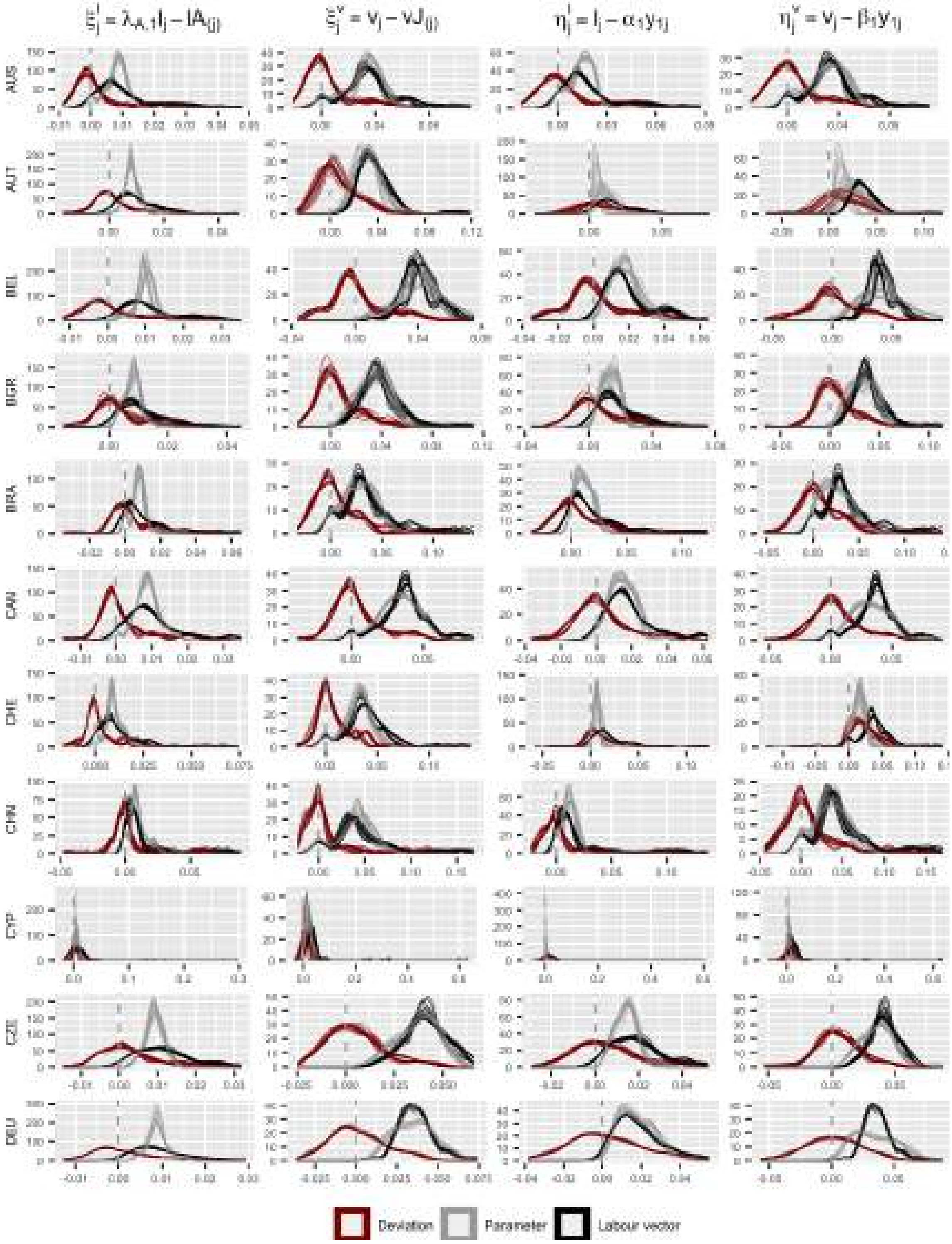


Figure D.16: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi_j^{\mathbf{l}}, \xi_j^{\mathbf{v}}, \eta_j^{\mathbf{l}}, \eta_j^{\mathbf{v}})$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

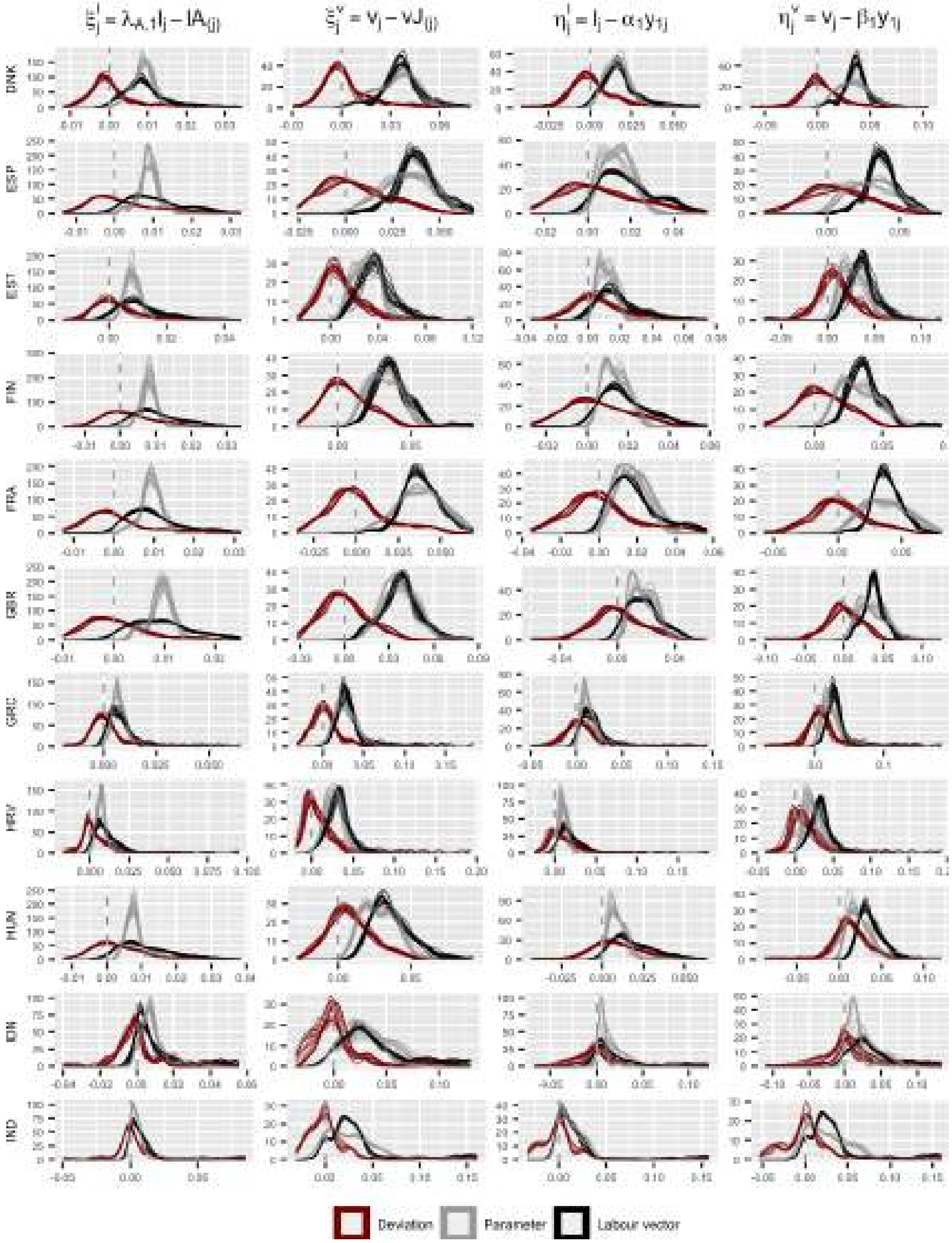


Figure D.17: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi_j^{\mathbf{l}}, \xi_j^{\mathbf{v}}, \eta_j^{\mathbf{l}}, \eta_j^{\mathbf{v}})$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

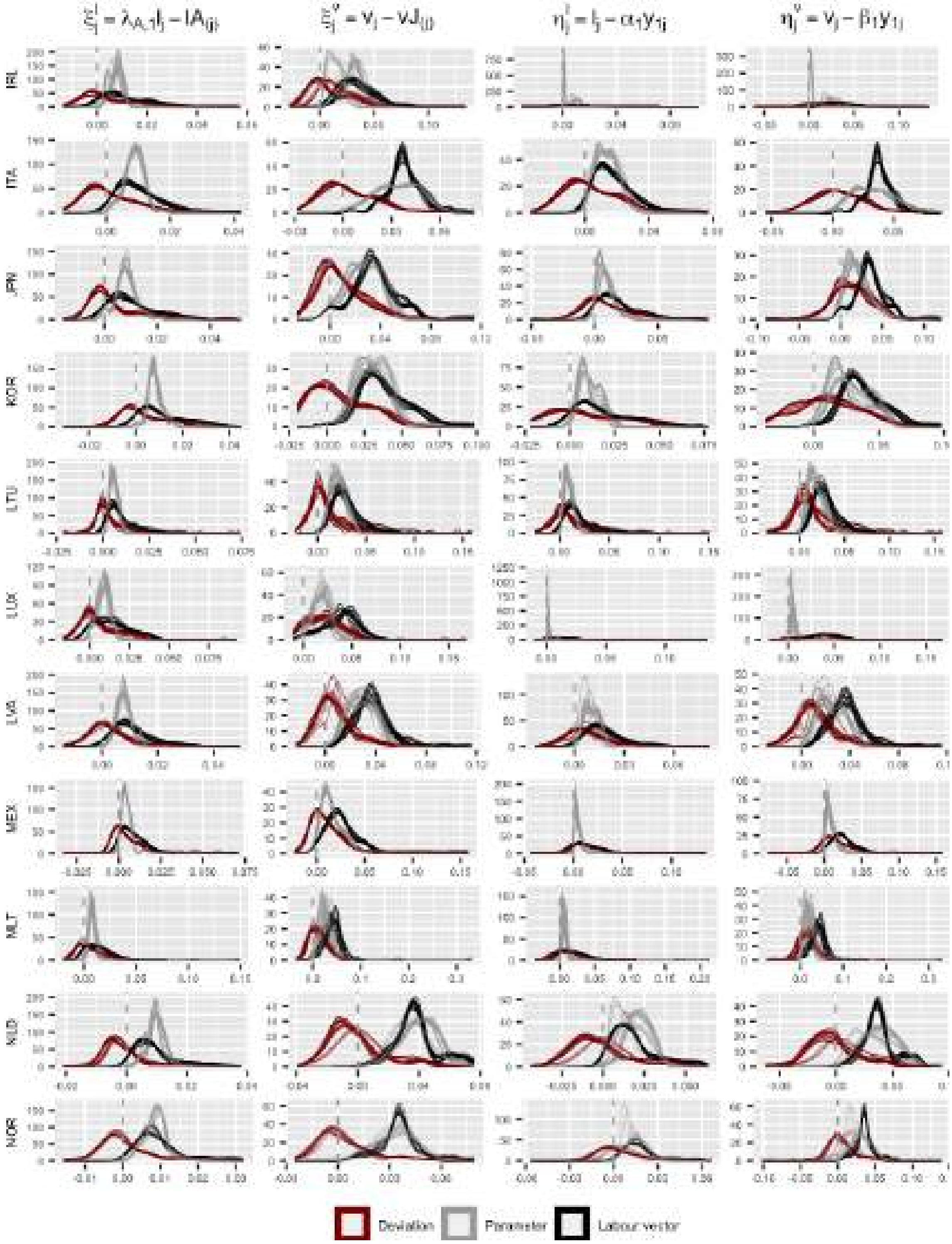


Figure D.18: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi^{\mathbf{l}}_j, \xi^{\mathbf{v}}_j, \eta^{\mathbf{l}}_j, \eta^{\mathbf{v}}_j)$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

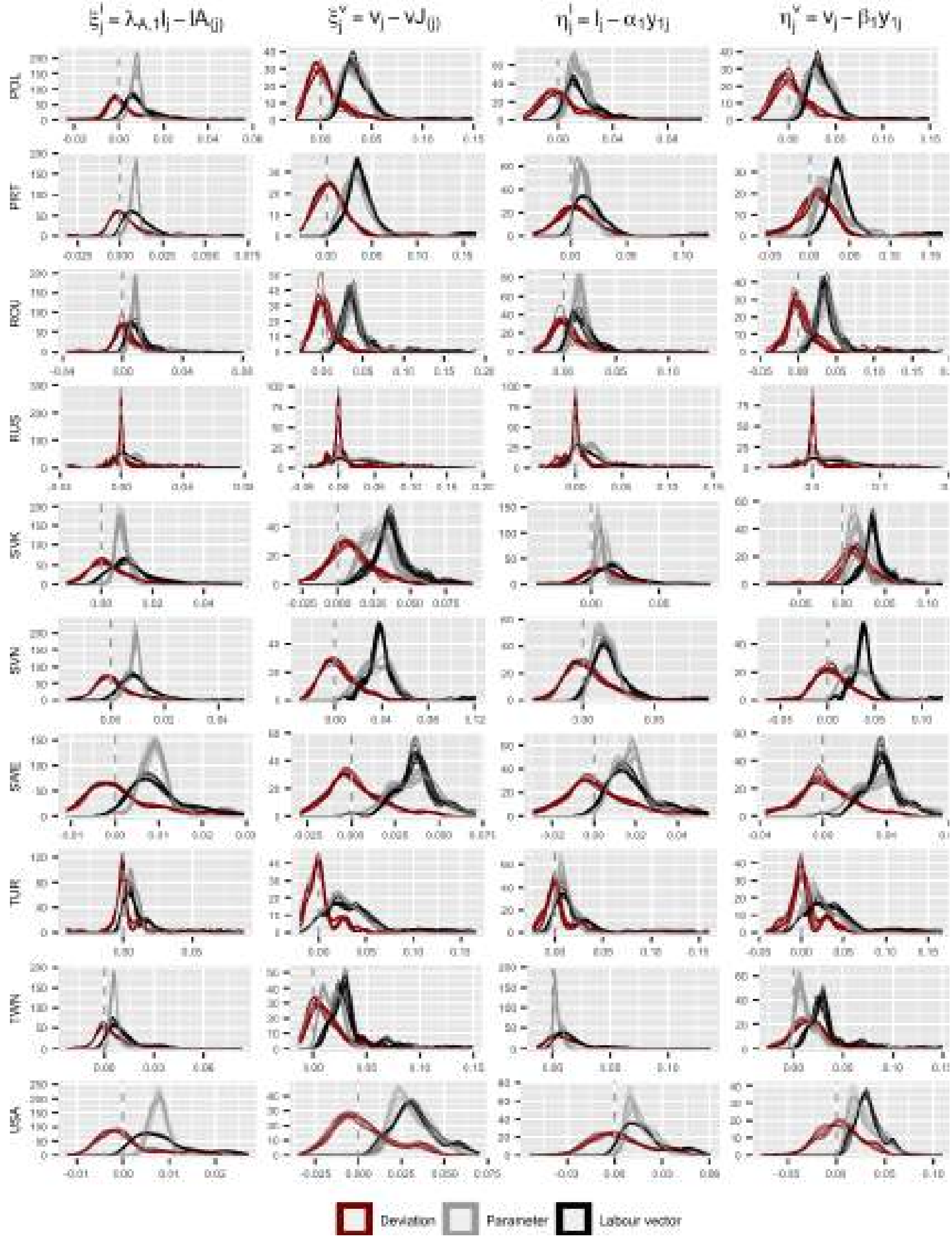


Figure D.19: Empirical densities of the coefficients from the three vectors defining the deviation: deviations, $(\xi_j^{\mathbf{l}}, \xi_j^{\mathbf{v}}, \eta_j^{\mathbf{l}}, \eta_j^{\mathbf{v}})$; labour vector, $(\lambda_{\mathbf{A}1} l_j, v_j, l_j, v_j)$; and parameters, $(\mathbf{lA}_{(j)}, \mathbf{vJ}_{(j)}, \alpha_1 y_{1j}, \beta_1 y_{1j})$. Source: authors' calculations based on the WIOD database, 2016 release.

E The spectral representation of the Mean Absolute Deviation (MAD) indicator of the distance between production prices $p_j(\rho)$ and embodied labour v_j.

The MAD is a scalar indicator widely used in the literature to assess the degree of closeness/proximity between production prices $p_j(\rho)$ of commodity j, computed at the observed profit rate in the economy, and its quantities of embodied labour v_j, often called labour values. Let $\chi_j \equiv p_j(\rho) - v_j$ be the difference between production prices and labour values of commodity j at the relative profit rate ρ. Then, the MAD is defined as

$$MAD(\boldsymbol{\chi}) \equiv \frac{1}{n} \sum\nolimits_{j=1}^{n} |\chi_j - \bar{\chi}| \,, \tag{E.1}$$

where $\mathbf{p}(\rho) - \mathbf{v} \equiv \boldsymbol{\chi} \equiv [\chi_j]$ is the $1 \times n$ vector of deviations between the vectors of production prices and labour values and $\bar{\chi} = \frac{1}{n} \sum_{j=1}^{n} \chi_j = \frac{1}{n} \boldsymbol{\chi} \mathbf{e}^T$ is the average of χ_j.

Equation (A.1) showed that production prices can be expressed as $\mathbf{p}(\rho) = \mathbf{v} + \rho \mathbf{f} + \cdots + \rho^{q_0} \mathbf{f}^{q_0} + \mathbf{d}_{q_0}$, where $\mathbf{d}_{q_0} \equiv \sum_{s=q_0+1}^{+\infty} \rho^s \mathbf{f}^s$. Hence, the vector of production prices-labour values differences equals

$$\boldsymbol{\chi} \equiv \mathbf{p}(r) - \mathbf{v} = \rho \mathbf{f} + \rho^2 \mathbf{f}^2 + \cdots = \sum\nolimits_{s=1}^{+\infty} \rho^s \mathbf{f}^s = \mathbf{d}_0, \tag{E.2}$$

where $q_0 = 0$. Equation (A.2) showed that $\mathbf{d}_{q_0}$ can be expressed as

$$\mathbf{d}_{q_0} = \rho^{q_0+1} \sum\nolimits_{k=2}^{n} \frac{\lambda_{\mathbf{J}k} - 1}{(1 - \rho \lambda_{\mathbf{J}k}) \mathbf{y}_k \mathbf{x}_k^T} \lambda_{\mathbf{J}k}^{q_0} \beta_k \mathbf{y}_k,$$

therefore constraining $q_0 = 0$ means that $\lambda_{\mathbf{J}k}^{q_0} = 1$, $\rho^{q_0+1} = \rho$, and

$$\mathbf{d}_0 = \sum\nolimits_{k=2}^{n} \frac{\rho\,(\lambda_{\mathbf{J}k} - 1)}{(1 - \rho \lambda_{\mathbf{J}k}) \mathbf{y}_k \mathbf{x}_k^T} \beta_k \mathbf{y}_k = \boldsymbol{\chi}. \tag{E.3}$$

Scalars $\frac{\rho(\lambda_{\mathbf{J}k} - 1)}{(1-\rho\lambda_{\mathbf{J}k}) \mathbf{y}_k \mathbf{x}_k^T} \beta_k$ multiply each coefficient $y_{k,j}$ of vector $\mathbf{y}_k$ so that

$$\chi_j = \sum\nolimits_{k=2}^{n} \frac{\rho\,(\lambda_{\mathbf{J}k} - 1)}{(1 - \rho \lambda_{\mathbf{J}k}) \mathbf{y}_k \mathbf{x}_k^T} \beta_k y_{k,j}. \tag{E.4}$$

Equations (E.3) and (E.4) tell us that if $r = 0$, then $\rho = 0$ and $\boldsymbol{\chi} = \mathbf{0} \rightarrow MAD(\boldsymbol{\chi}) = 0$. We can get $\boldsymbol{\chi} = \mathbf{0}$ for every profit rate $r \in [0, R]$ (for every $\rho \in [0, 1]$) if and only if $\beta_{k \geq 2} = 0$.[E.1] We cannot achieve the same result relying on the subdominant eigenvalues $\lambda_{\mathbf{J}k}$ and the eigenvectors $\mathbf{y}_k$ —We know that (1)

[E.1] We can achieve this if and only if the labour coefficient vector is proportional to the left-hand Perron-Frobenius vector. This is the case of equal capital intensities across industries.

primitivity of matrix $\mathbf{J}$ implies that $|\lambda_{\mathbf{J}k\geq 2}| < 1$ and that (2) eigenvectors $\mathbf{y}_k$ cannot be null vectors.

Let us proceed deriving the spectral representation of the $MAD(\boldsymbol{\chi})$ in order to identify which factors can make this scalar indicator small. Define the mean value of the coefficients within eigenvector $\mathbf{y}_k$ as $\bar{y}_k \equiv \frac{1}{n}\sum_{j=1}^{n} y_{k,j} = \frac{1}{n}\mathbf{y}_k\mathbf{e}^T$. Then,

$$
\begin{aligned}
\bar{\chi} = \frac{1}{n}\left[\mathbf{d}_0\mathbf{e}^T\right] &= \frac{1}{n}\left[\sum\nolimits_{k=2}^{n} \frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k\mathbf{y}_k\mathbf{e}^T\right] \\
&= \sum\nolimits_{k=2}^{n} \frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k\bar{y}_k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\chi_j - \bar{\chi} &= \sum\nolimits_{k=2}^{n} \frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k y_{k,j} - \sum\nolimits_{k=2}^{n} \frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k\bar{y}_k \\
&= \sum\nolimits_{k=2}^{n} \frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k\left[y_{k,j}-\bar{y}_k\right]. \qquad \text{(E.5)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
MAD(\boldsymbol{\chi}) &\equiv \frac{1}{n}\sum\nolimits_{j=1}^{n} |\chi_j - \bar{\chi}| \\
&= \frac{1}{n}\sum\nolimits_{j=1}^{n} \left|\sum\nolimits_{k=2}^{n} \frac{\rho\left(\lambda_{\mathbf{J}k}-1\right)}{(1-\rho\lambda_{\mathbf{J}k})\mathbf{y}_k\mathbf{x}_k^T}\beta_k\left[y_{k,j}-\bar{y}_k\right]\right|. \qquad (19)
\end{aligned}
$$

Based on the multiplicativity and subadditivity properties of the absolute value, we can rearrange the sums in (19) and provide an upper bound for the $MAD(\boldsymbol{\chi})$ given by

$$
\begin{aligned}
MAD(\boldsymbol{\chi}) &\leq \sum\nolimits_{k=2}^{n} |\rho| \left|\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})}\right| \left|\frac{\beta_k}{\mathbf{y}_k\mathbf{x}_k^T}\right| \frac{\sum_{j=1}^{n}|y_{k,j}-\bar{y}_k|}{n} \\
&= \sum\nolimits_{k=2}^{n} |\rho| \left|\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})}\right| \left|\frac{\beta_k}{\mathbf{y}_k\mathbf{x}_k^T}\right| \cdot MAD(\mathbf{y}_k) \qquad (20)
\end{aligned}
$$

Equations (19) and (20) show different ways in which we can achieve a zero or sufficiently small $MAD(\boldsymbol{\chi})$:

(1) It seems possible to have a $0 \leq r = \rho R \leq R$ as small as we want to achieve a desired small $MAD(\boldsymbol{\chi}) \geq 0$. This constraint has already been identified in literature (e.g., Mariolis and Tsoulfidis, 2010)

(2) We can also reduce the subdominant eigenlabours $\beta_{k\geq 2}$ as much as we want to obtain a prescribed maximum $MAD(\boldsymbol{\chi}) \geq 0$, but this time independently of the rate of profits.

(3) Small variability of the coefficients of the eigenvectors $[y_{k,j} - \bar{y}_k]$ and, in particular, small $MAD(\mathbf{y}_k)$ for $k = 2, \ldots, n$ can also reduce the magnitude of

the MAD distance.[E.2] Just like in the identification of the characteristics of the productive structure behind the regularities in price curves (see the introduction to section 3), it is left for future research the identification of the statistical characteristics of the eigenvectors in observed economies.

(4) The influence of the terms $\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})}$ on the $MAD(\boldsymbol{\chi}) \geq 0$ depends on the interaction between the relative profit rate ρ and the subdominant eigenvalues $\lambda_{\mathbf{J}k\geq 2}$. For every economy under study, most of $\lambda_{\mathbf{J}k\geq 2}$ concentrate around zero. Hence, for these sufficiently small subdominant $\lambda_{\mathbf{J}k}$, the terms $\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})} \approx -1$. Given that $\rho \in [0, 1)$, this result seems to be robust to any admissible ρ. We know turn to the $\lambda_{\mathbf{J}k\geq 2}$ with the highest magnitude (for some economies, $\lambda_{\mathbf{J}2} \approx 0.5$). Suppose that these $\lambda_{\mathbf{J}k}$ are real. On the one hand, as $\rho \to 1$ the term $\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})} \to -1$, independently of $\lambda_{\mathbf{J}k}$. On the other hand, as $\rho \to 0$ then $\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})} \to \lambda_{\mathbf{J}k} - 1$. The fact that a great number of $\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})} \approx -1$ and a few $\frac{(\lambda_{\mathbf{J}k}-1)}{(1-\rho\lambda_{\mathbf{J}k})}$ are different from zero (for real eigenvalues, between -1 and -0.5), the persistent small values of $MAD(\boldsymbol{\chi})$ across economies seems not to be related with the subdominant eigenvalues.

[E.2] Given that $\beta_{k\geq 2} = 0$ are both necessary and sufficient conditions for $\boldsymbol{\chi} = \mathbf{0}$, it might not be possible to have $MAD(\mathbf{y}_k) = 0$ for $k = 2, \ldots, n$. Hence, eigenvectors might need to display some variability.

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