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Christian Bidard

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Abstract
We fully characterise, in terms of their local or global properties, the wage curves associated with Austrian models of production. When these properties are met, the degrees of freedom in the choice of parameters allow us to build an Austrian model which admits a given wage curve and satisfies other requirements.

Keywords: Austrian model, long run, trade-off property, wage curve

JEL Codes: B53, D24, D33

1. Introduction

The[1] trade-off property between wages and profits means that, in long-run equilibria, the wage curve is a decreasing function of the rate of profit (Ricardo 1817; Sraffa 1960). This, however, is one of the rare properties common to all multisector models. Suppose we do not know the underlying multisector model but observe the behaviour of the real wage and of one or several other magnitudes such as the capital-output ratio (or its inverse, the productivity of capital), either over a certain range or for all rates of profit. We would like to mimic these observations and reproduce them in a simple model. In general, a neoclassical model will not fit because its capital-output ratio varies monotonously with the rate of profit, also because the real wage is a convex function of the rate of profit, and these are too specific properties. An alternative simple candidate is an Austrian model: production is then represented by a flow of dated labour inputs which produces one unit of final good (Böhm-Bawerk 1889; Wicksell 1901; Hicks 1939). It is unlikely that the observations can be reproduced by means of a given Austrian method of production. But a family of Austrian methods depending on one or several parameters may have a rather complex behaviour (for instance, reswitching is not excluded), and this may be a good practical compromise between simplicity and flexibility. Simplicity comes from the hypothesis of a unique final good and the lack of interindustrial relationships (capital is the present

[1] With acknowledgements to an anonymous referee.

value of past quantities of labour over a finite number of periods), flexibility is introduced by leaving some leeway in the choice of the intertemporal process: with a continuum of Austrian processes, the operated method, which is characterised by its cost-minimisation or wage-maximisation property, depends on distribution.

Austrian models also have specific properties and curves which do not satisfy them cannot be mimicked by such a model. For instance, the wage is positive for any rate of profit in an Austrian model, whereas it may vanish for some finite rate of profit in multisector models with physical inputs. This, however, is not an objection if the range of observation of the real magnitudes is bounded. To characterise the properties which can be reproduced by means of an Austrian model amounts to identifying the specific properties of that type of models. We provide a complete answer to that question for the wage curve, but the tool we use may be adapted to other problems. The tool itself is introduced in Section 2: each Austrian process within a continuum being identified by a certain parameter, a change in the name of the parameter has clearly no influence on the economic properties. The idea is to reparameterise the initial family in a way which eases further calculations. In Section 3, the tool is first applied to two-period Austrian models. A complete characterisation of wage curves is obtained: we first find some properties of the wage curves, then show that these properties are exhaustive. Moreover, these properties are global, i.e. they concern the whole curve even if they also admit a local interpretation (such as convexity, which may be seen either as a global or a local property). It is known, however, that two-period Austrian models are rather peculiar and behave in many respects like neoclassical models (past labour may be seen as aggregate capital). T-period models are richer, in the sense that they serve as a support for more involved wage curves, which means that their wage curves share less common properties. In Section 4, we identify the local properties of the wage curves. The distinction between local and global properties comes from calculus, as the first order condition associated with the cost-minimisation problem expresses a local property. However, the proof leads attention to a specific type of T-period models, those for which labour is invested only at the initial and the final dates. We call them Sekt economies because the wine industry is sometimes quoted as an example of production with no intermediate labour investment. Section 5 provides a global characterisation of wage curves in Sekt economies. For the local but simultaneous reproduction of several curves (e.g., the wage curve and the capital-output ratio), T-period models remain worth of attention, because they admit many degrees of freedom in the choice of the labour inputs once the problem has been solved for the wage curve. These degrees of freedom can be used to meet other constraints.

For a brief overview avoiding technical details, the properties of wage curves are stated in Definition 2, and Theorems 3 and 4 constitute the main economic results of the paper.

2. The normal parameterisation

An Austrian T-period process is formalised as a sequence of dated past labour inputs $(l_{T-1}, l_{T-2}, ..., l_1, l_0)$ which produces one unit of final good at date 0. The time index in the labour vector refers to the date at which labour is paid: with wages paid *post factum*, l_{T-1} is the amount of labour invested at date $-T$ and l_0 the amount invested at date -1, so that the production process does take T periods. For a given process no substitution is possible between past and present labour; some substitution occurs in the presence of a continuum of processes, when the labour coefficients l_i depend on a real parameter s. Then the operated process at a given rate of profit r is the one which minimises the overall cost of production, i.e. it is the solution $s = s(r)$ of

$$\min_s \sum_i (1+r)^i l_i(s) \tag{1}$$

By competition between entrepreneurs, that minimum value $v(r)$ is the inverse $w(r)^{-1}$ of the real wage in terms of final good.

Consider for example the 2-period family ($T = 2$) in which method s is defined by formulas

$$\begin{aligned} l_0(s) &= -s^2 - 2s + 24 \\ l_1(s) &= s^2 \end{aligned}$$

for s varying in the interval $[0, 4]$. By setting $u = \sqrt{s}$, the same family is described as

$$\begin{aligned} l_0(u) &= -u^4 - 2u^2 + 24 \\ l_1(u) &= u^4 \end{aligned}$$

for $u \in [0, 2]$. Clearly enough, the replacement of s by u leads to a new parameterisation of the family but does not affect it. One may wonder if some parameterisation is more fruitful than others for theoretical purposes. The one we propose attributes to each method a label t which coincides with the rate of profit at which that method is cost-minimising. Let us show how it works for the above s-family. Since the overall cost of production for method s amounts to $(1+r)l_1(s) + l_0(s)$, calculus shows that the minimum is reached for the method with label $s = r^{-1}$ (at least if $r \geq 0.25$). Let us set $t = s^{-1}$. The above s-family of methods is rewritten as a t-family

$$\begin{aligned} l_0(t) &= \frac{24t^2 - 2t - 1}{t^2} \\ l_1(t) &= \frac{1}{t^2} \end{aligned}$$

($t \geq 0.25$) and, by construction, the cost-minimising method at rate r in the t-parameterisation is the one with label $t = r$: we dub it the 'normal' parameterisation.

Three comments are in order:

- First, we ignore corner solutions, so that optimisation problems are treated by means of standard calculus rather than by the Kuhn and Tucker conditions. (For the family here considered, the cost-minimising method for a rate of profit smaller than 0.25 is the one with index $t = 0.25$.) We also ignore the case when more than one method is cost-minimising at some rate of profit.

- Second, the normal (re)parameterisation may lead to an impoverishment of the set of methods: those which are never used receive no label and are ignored. This is not a problem since they play no effective role. Conversely, a method which is operated at different rates of profit (reswitching) receives several labels. This, again, is not a problem.

- Third, the reparameterisation works for any family of Austrian methods, more generally for any family of methods.

From a theoretical point of view, the reparameterisation exercise is unnecessary: given a family of methods, we assign label $t = r$ to the cost-minimising method at the rate of profit r. Then, by construction and from the very beginning, the t-parameterisation has the required property. There remains to show its usefulness. (We expect that the identification of the value t of the generic parameter and that r of the rate of profit, which characterises the normal parameterisation, will not be a source of confusion in the following calculations: when a property holds for any parameter, the name given to that parameter does not matter.)

3. The wage curve in two-period economies

The aim of that Section is to characterise the wage curves $w = w(r)$ of two-period economies. As it turns out that it is simpler to deal with the inverse of the real wage than with the wage itself, we set $v = w^{-1}$ (we call it a $v-$curve) and look first at the properties of that curve.

Since we use the normal parameterisation, method r is operated at rate r, and therefore we have identity

$$v(r) = l_0(r) + (1+r)l_1(r) = \min_t (l_0(t) + (1+r)l_1(t)) \tag{2}$$

It follows from the first order conditions on t that equality

$$l_0'(r) + (1+r)l_1'(r) = 0 \tag{3}$$

holds for any r. As a consequence, the derivative of $v(r) = l_0(r) + (1+r)l_1(r)$ is equal to $l_1(r)$

$$\begin{aligned} l_1(r) &= v'(r) & (4)\\ l_0(r) &= v(r) - (1+r)v'(r) & (5) \end{aligned}$$

We look for properties of the curve $v(r)$. Equalities (4) and (5) show that $v'(r)$ and $v(r) - (1+r)v'(r)$ are nonnegative. Are there other properties? A known property of two-period Austrian models is that a rise in the rate of interest leads to the substitution of present labour for past labour (the property will be proved independently as a consequence of relationships (6) below) and, therefore, $v'(r) = l_1(r)$ is a decreasing function, i.e. function v is concave. Can other properties be found? A suggestion might be that the wage tends to zero when the rate of profit tends to infinity, therefore that $v(r) = w^{-1}(r)$ goes to infinity with r. This, however, is not always the case. To check the exhaustiveness of the above three properties, we consider the inverse problem: that is, we start from a given curve $\alpha(r)$ satisfying the three properties. If one can build a two-period economy $(l_1(r), l_0(r))$ whose curve $v(r) = w^{-1}(r)$ coincides with $\alpha(r)$, then the v-curves have no other general property. In the inverse problem, the potential $v-$curve is given and functions $l_0(r)$ and $l_1(r)$ are the unknowns.

Lemma 1 *A given curve $v = v(r)$ is the $v-$curve of a two-period economy if and only if:*
(i) it is positive, increasing and concave,
(ii) function $(1+r)^{-1}v(r)$ is decreasing.
The corresponding economy is uniquely defined by its $v-$curve.

Proof. Since the decreasingness of $(1+r)^{-1}v(r)$ amounts to condition $v(r) - (1 + r)v'(r) \geq 0$, it has been shown above that properties (i) and (ii) are necessary. Conversely, consider a function v satisfying the properties stated in Lemma 1. We define functions l_1 and l_0 by means of relations (4) and (5). That is, we introduce the nonnegative functions $l_1(t) = v'(t)$ and $l_0(t) = v(t) - (1+t)v'(t)$ and consider the Austrian economy characterised by the methods ($l_1(t)$, $l_0(t)$).

Since $l_0(t) + (1+t)l_1(t) = v(t)$ by construction, derivation shows that $l_0'(t) + (1 + t)l_1'(t) = v'(t) - l_1(t) = 0$. In the Austrian economy ($l_1(t)$, $l_0(t)$), the choice of the cost-minimising method at rate r leads us to minimise the expression $\Delta(t) = (1+r)l_1(t) + l_0(t)$. The derivative of that function is $\delta(t) = (1+r)l_1'(t) + l_0'(t) = (r-t)l_1'(t) = (r - t)v''(t)$. According to the concavity hypothesis on v, function $\delta(t)$ is negative for $t < r$, vanishes at $t = r$ and is positive for $t > r$. Therefore the minimum of the cost function is reached for method $t = r$. In that economy, the inverse wage ω^{-1} at rate r amounts to $\omega^{-1}(r) = (1+r)l_1(r) + l_0(r) = v(r)$. To sum up, starting from an arbitrary curve with the properties mentioned in the Lemma, we have built an Austrian economy which admits it as its $v-$curve. The construction also shows uniqueness (up to the introduction of methods which are never operated), the characteristics of the economy being defined by equalities (4) and (5). ■

Remark. Let condition (i) be met. We have noticed that condition (ii) is equivalent to inequality $v(r) - (1+r)v'(r) \geq 0$. Since function $v(r) - (1+r)v'(r)$ is increasing by the concavity property of v, its positivity is ensured for any rate of interest if and only if it holds at $r = 0$. Therefore condtion (ii) can be replaced by the equivalent condition $v(0) \geq v'(0)$.

Lemma 1 identifies the $v-$curves of two-period economies, with $v(r) = w(r)^{-1}$. To characterise the wage curves, it suffices to translate these properties in terms of function w. For instance, the property '$v(r)$ is increasing and $(1+r)^{-1}v(r)$ is decreasing' means that the wage $w(r)$ is decreasing but $(1+r)w(r)$ is increasing, i.e. the real wage is decreasing with the rate of profit (Ricardian trade-off) but not too much. To check that result, let us prove it independently of the differentiability hypothesis.

Lemma 2 *In a two-period Austrian model, the real wage $w(r)$ is decreasing with the rate of profit and $(1+r)w(r)$ is increasing.*

Proof. Let (l_0, l_1) be the cost-minimising method at rate r, and $(\overline{l_0}, \overline{l_1})$=$(l_0+\Delta l_0, l_1+\Delta l_1)$

that at rate $\overline{r}$ with $\overline{r} > r$. Cost-minimisation is expressed by inequalities

$$\begin{aligned} w^{-1}(r) &= l_0 + (1+r)l_1 \leq (l_0 + \Delta l_0) + (1+r)(l_1 + \Delta l_1) \\ w^{-1}(\overline{r}) &= (l_0 + \Delta l_0) + (1+\overline{r})(l_1 + \Delta l_1) \leq l_0 + (1+\overline{r})l_1 \end{aligned}$$

Hence, more compactly:

$$\Delta l_0 + (1+\overline{r})\Delta l_1 \leq 0 \leq \Delta l_0 + (1+r)\Delta l_1 \tag{6}$$

The inequalities $w^{-1}(r) \leq w^{-1}(\overline{r})$ and $(1+r)w^{-1}(\overline{r}) \leq (1+\overline{r})w^{-1}(r)$ we want to establish are written:

$$l_0 + (1+r)(\overline{l_1} - \Delta l_1) \leq (l_0 + \Delta l_0) + (1+\overline{r})\overline{l_1} \tag{7}$$

$$(1+r)(l_0 + \Delta l_0 + (1+\overline{r})\overline{l_1}) \leq (1+\overline{r})(l_0 + (1+r)(\overline{l_1} - \Delta l_1)) \tag{8}$$

Consider both sides of inequality (7) as affine functions of $\overline{l_1}$. As the inequality holds for high values of $\overline{l_1}$ (because $\overline{r} > r$) and for $\overline{l_1} = 0$ (by condition (6)), it holds in any case. The same arguments applies to inequality (8) when its both sides are considered as affine functions of l_0. Therefore properties (7) and (8) do hold. ■

Relation (6) with $\overline{r} > r$ implies that Δl_0 is positive and Δl_1 negative, a property temporarily admitted above.

Theorem 1 *A curve $w = w(r)$ is the wage curve of a two-period economy if and only if it is positive and decreasing, function $(1+r)w(r)$ is increasing and*

$$2w'^2 \leq ww'' \tag{9}$$

The economy is uniquely defined by its wage curve.

Proof. Since $v = w^{-1}$, we have $v' = -w'w^{-2}$ and $v'' = (2w'^2 - ww'')w^{-3}$. Theorem 1 is a rewriting of Lemma 1 in terms of the wage curve. ■

Remark. Let us assume inequality (9). The derivative of function $(w + (1+r)w')w^{-2}$ being $(1+r)w^{-3}(ww'' - 2w'^2) \geq 0$, function $w + (1+r)w'$ is positive for any r if and only if it is positive at $r = 0$. Therefore, the global condition 'function $(1+r)w(r)$ is increasing' in Theorem 1 can be replaced by the initial condition $w(0) + w'(0) \geq 0$.

Thanks to the simplifications introduced by the normal parameterisation, the main point of the above proof consists in showing that the first order condition leads to a global minimum.

4. *T*-period economies

In that Section, we generalise Theorem 1 and identify the wage curves in *T*-period Austrian economies by their properties. The argument follows the same general line of proof, but some technical difficulties lead us to state a local result. Lemmas 3, 4 and Theorem 2 first characterise the properties of curves $v(r) = w^{-1}(r)$.

Lemma 3 *Let $w = w(r)$ be a given function of the rate of interest r defined on a small interval and let $v(r) = w^{-1}(r)$. The curve $w(r)$ coincides locally with the wage curve of a T-period economy if and only if there exist nonnegative functions ($l_0(t), ..., l_{T-1}(t)$) satisfying the conditions*

$$\sum_{i=0}^{T-1}(1+t)^i l_i(t) = v(t) \tag{10}$$

$$\sum_{i=0}^{T-1}(1+t)^i l_i'(t) = 0 \tag{11}$$

$$\sum_{i=1}^{T-1} i(1+t)^{i-1} l_i'(t) \geq 0 \tag{12}$$

Proof. Given a family of *T*-period economies which admits $v = v(r)$ as its v-curve, we reparameterise it and adopt the normal parameterisation ($l_0(t), ..., l_{T-1}(t)$) of that family. This once done, condition (10) expresses that $v(t)$ is the inverse of the wage at rate t. Conditions (11) and (12) express that, in a neighbourhood of $t = r$, the marginal cost function $\sum_{i=0}^{T-1}(1+r)^i l_i'(t)$ is negative for $t \leq r$, zero at $t = r$ and positive for $t \geq r$. Hence, the functions $l_i(t)$ satisfy the required conditions.

Conversely, let a given function $v(t)$ for which there exist functions $l_i(t)$ such that conditions (10)-(11)-(12) hold. Consider the Austrian economy attached to these functions $l_i(t)$. The cost function (with labour as numeraire) being $\Delta(t) = \sum_{i=0}^{T-1}(1+r)^i l_i(t)$, its derivative $\delta(t)$ in a neighbourhood of $t = r$ is

$$\begin{aligned}
\delta(t) &= \sum_{i=0}^{T-1}(1+r)^i l_i'(t) \\
&= \sum_{i=0}^{T-1}((1+r)^i - (1+t)^i) l_i'(t) \\
&= (r-t)\sum_{i=1}^{T-1} i(1+r)^{i-1} l_i'(t) + \varepsilon(t-r)
\end{aligned}$$

It follows from condition (12) that, in a neighbourhood of r, $\delta(t)$ is negative for $t \leq r$, vanishes at $t = r$ and is positive for $t \geq r$, therefore the cost function admits a local minimum at $t = r$. Equality (10) shows that it is then equal to $v(r)$. The wage function is therefore $v^{-1}(r) = w(r)$. ■

Comparing Lemma 3 with Theorem 1 lets appear two differences: first, Lemma 3 is local whereas Theorem 1 is global; and, second, if the system (10)-(11)-(12) has a solution for $T > 2$, it admits infinitely many solutions, because there are T degrees of freedom in the choice of the unknown functions $l_i(t)$ and only two binding constraints.

Lemma 3 reduces the characterisation of the wage curves to an algebraic problem, which consists in identifying the functions v for which the system (10)-(11)-(12) admits a nonnegative solution. The next Lemma proposes another statement of the same problem.

Lemma 4 *Let $u = (T-1)v - (1+t)v'$. The system (10)-(11)-(12) is equivalent to the system*

$$\sum_{i=0}^{T-2}(T-1-i)(1+t)^i l_i(t) = u(t) \tag{13}$$

$$\sum_{i=1}^{T-1} i(1+t)^{i-1} l_i(t) = v'(t) \tag{14}$$

$$\sum_{i=1}^{T-2} i(T-1-i)(1+t)^{i-1} l_i(t) \leq u' \tag{15}$$

Proof. Let us first assume equality (10). Calculating the derivative of v defined by (10) shows the equivalence of equalities (11) and (14). For v defined by (10) and v' by (14), function $u = (T-1)v - (1+t)v'$ is equal to $u = \sum_{i=0}^{T-1}(T-1-i)(1+t)^i l_i(t)$, therefore

$$u' = \sum_{i=1}^{T-2} i(T-1-i)(1+t)^{i-1} l_i(t) + (T-1)\sum_{i=0}^{T-1}(1+t)^i l_i'(t) - (1+t)\sum_{i=1}^{T-1} i(1+t)^{i-1} l_i'(t)$$

where the second sum is zero by (11). Hence the equivalence between inequalities (12) and (15). The partial conclusion is the equivalence between systems (10)-(11)-(12) and (10)-(14)-(15). When (14) holds, we can replace (10) by (13), which is the equality obtained by eliminating l_{T-1} between (10) and (14). This shows the equivalence between systems (10)-(11)-(12) and (13)-(14)-(15). ■

The peculiarity of the transformed system is that $l_0(t)$ appears in equation (13) only and $l_{T-1}(t)$ in equation (14) only. The following Lemma shows that a property we will refer to can take different forms.

Lemma 5 *Let $v(r)$ be a differentiable real function defined for $r \geq 0$. The decreasingness of function $(1+r)^{1-T}v(r)$ is equivalent to inequality $(T-1)v - (1+r)v' \geq 0$, and the following three properties (16), (17) and (18) are also equivalent:*

$$\begin{aligned} u(r) &= (T-1)v(r) - (1+r)v'(r) \text{ is increasing} & (16)\\ &\Leftrightarrow (1+r)^{2-T}v' \text{ is decreasing} & (17)\\ &\Leftrightarrow (2-T)v' - (1+r)v" \leq 0 & (18)\end{aligned}$$

Similarly for the following three properties relative to a differentiable function $w(r)$ defined for $r \geq 0$:

$$\begin{aligned} & [(T-1)w(r) + (1+r)w'(r)]\, w^{-2} \text{ is decreasing} & (19)\\ \Leftrightarrow\ & (1+r)^{2-T}w^{-1}w' \text{ is increasing} & (20)\\ \Leftrightarrow\ & (2-T)ww' + (1+r)(ww" - 2w'^2) \geq 0 & (21)\end{aligned}$$

Proof. Immediate from calculus. ■

Inequality (21) generalises inequality (9) to the case $T > 2$.

Definition 1 *A positive function $v = v(r)$ has property (V_T) if and only if:*

(i) function $v(r)$ is increasing,

(ii) function $(1+r)^{1-T}v(r)$ is decreasing,

(iii) function $(1+r)^{2-T}v'(r)$ is decreasing.

Theorem 2 *Let $v = w^{-1}$. A positive curve is locally the $v-$curve of a T-period economy if and only if property (V_T) is met locally.*

Proof. Lemmas 3 and 4 link the existence of a T-period Austrian economy sustaining locally a given v-curve to the existence of a nonnegative solution $l_0(t), ..., l_{T-1}(t)$ of system (13)-(14)-(15). Conditions $v' \geq 0, u \geq 0$ and $u' \geq 0$ are clearly necessary. Conversely, if these conditions hold, one can choose functions $l_1(t), ..., l_{T-2}(t)$ which are nonnegative but small enough to meet inequality (15) and to admit a nonnegative solution $l_0(t)$ of equation (13) and a nonnegative solution $l_{T-1}(t)$ of equation (14). Therefore, inequalities

$v' \geq 0, u \geq 0$ and $u' \geq 0$ characterise the v-curves. The first inequality is equivalent to condition (i) in Definition 1, the second to condition (ii) (because the derivative of $(1+r)^{1-T}v(r)$ has the sign of $-u$) and the third to condition (iii) (by Lemma 5). ■

The last proof confirms the existence of infinitely many solutions when *T* is greater than two (at least when inequalities (15) or (18) are strict): infinitely many Austrian models generate locally the same $v-$curve. The local existence property, combined with multiplicity, is a sufficient result when the aim is to mimic locally the behaviour of both a given wage curve and another magnitude, say the capital-output ratio. Then, thanks to the degrees of freedom in the choices of the intermediate functions $l_1(t), ..., l_{T-2}(t)$, one may expect to build an Austrian model with the required properties.

There remains to translate the above properties in terms of the wage curve itself.

Definition 2 *A positive function $w = w(r)$ has property (W_T) if:*
(i) $w(r)$ is decreasing,
(ii) $(1+r)^{T-1}w(r)$ is increasing,
(iii) $(1+r)^{T-2}w^2(-w')^{-1}$ is increasing.

Theorem 3 *A positive curve is locally the wage curve $w(r)$ of a T-period economy if and only if property (W_T) is met locally.*

Proof. This a reinterpretation of Theorem 2 with $v = w^{-1}$. ■

Remark. Condition (ii) in Definition 1 means that function u is positive and, by Lemma 5, condition (iii) means that u is increasing. Therefore, under condition (iii), condition (ii) holds on some interval if it holds at the beginning of the interval. In particular, condition (ii) always holds if and only if $u(0) \geq 0$, i.e. if and only if $(T-1)v(0) \geq v'(0)$. Similarly, condition (ii) in Definition 2 holds globally if and only if $(T-1)w(0)+w'(0) \geq 0$.

5. Sekt economies

Theorem 3 generalises Theorem 1 but states a local existence result whereas, in the specific case *T* = 2, Theorem 1 was global. Can a global existence result also be found for *T* greater than two? Scrutinizing the proof of Theorem 2 lets appear that the degrees of freedom concern intermediate functions which are nonnegative but small. Attention is thus drawn to the specific solution for which these functions are zero: $l_1(t) = ... = l_{T-2}(t) = 0$. In that type of Austrian economy, labour investments occur at the initial and final dates only. Let us call it a Sekt economy (even if it is unlikely that present labour can be substituted for past labour in the wine industry!). A Sekt economy is analogous to a two-period economy of the type studied in Section 3 with a change in the length of the period: its qualitative behaviour is identical but the formulas must be adapted since a factor of interest $1+r$ per period corresponds to a factor $1+r_{T-1}=(1+r)^{T-1}$ between dates 0 and $T-1$.

Let us adopt the normal parameterisation and start calculations afresh. We already know from Section 3 that an increase in the rate of profit leads to the substitution of present labour for past labour, and therefore $l'_{T-1}(r)\leq 0$. Given the wage curve $w=w(r)$ or, better, $v(r)=w^{-1}(r)$, we have

$$\min_t l_0(t)+(1+r)^{T-1}l_{T-1}(t)=v(r) \tag{22}$$

For the normal parameterisation, the minimum is reached at $t=r$:

$$\begin{aligned} l_0(r)+(1+r)^{T-1}l_{T-1}(r) &= v(r) \qquad (23)\\ l'_0(r)+(1+r)^{T-1}l'_{T-1}(r) &= 0 \qquad (24)\end{aligned}$$

A comparison of the derivative of v as given by (23) with (24) leads us to identity

$$(T-1)(1+r)^{T-2}l_{T-1}(r)=v'(r) \tag{25}$$

Then explicit formulas for l_0 and l_{T-1} result from (23) and (25):

$$\begin{aligned} l_0(r) &= v(r)-\frac{1}{T-1}(1+r)v'(r) \qquad (26)\\ l_{T-1}(r) &= \frac{1}{T-1}(1+r)^{2-T}v'(r) \qquad (27)\end{aligned}$$

The partial conclusion is that the v-curve of a Sekt economy must be such that

$$\begin{aligned} v(r)-\frac{1}{T-1}(1+r)v'(r) &\geq 0 \qquad (28)\\ v'(r) &\geq 0 \qquad (29)\\ \frac{d}{dr}[(1+r)^{2-T}v'(r)] &\leq 0 \qquad (30)\end{aligned}$$

Lemma 6 *A curve $v(r)$ is the $v-$curve of a Sekt economy if and only if properties (28)-(29)-(30) hold.*

Proof. There remains to show that these conditions are sufficient. Let there be a curve $v=v(t)$ for which these conditions hold and consider the economy defined by data (26) and (27). At rate r, the real wage $w(r)$ is such that $w^{-1}(r)=\min_t \Delta(t)$, where $\Delta(t)=l_0(t)+(1+r)^{T-1}l_{T-1}(t)$. We get from formulas (26) and (27) that $l_0(t)+(1+t)^{T-1}l_1(t)=v(t)$, hence by derivation identity $l'_0(t)+(1+t)^{T-1}l'_{T-1}(t)=v'(t)-(T-1)(1+t)^{T-2}l_1(t)=0$.

Therefore the derivative $\delta(t)$ of $\Delta(t)$ amounts to

$$\begin{aligned}\delta(t) &= l_0'(t) + (1+r)^{T-1} l_{T-1}'(t) \\ &= [(1+r)^{T-1} - (1+t)^{T-1}] l_{T-1}'(t) \\ &= [(1+r)^{T-1} - (1+t)^{T-1}] \frac{1}{T-1} \frac{d}{dt} [(1+t)^{2-T} v'(t)]\end{aligned}$$

Inequality (30) defines the sign of $\delta(t) \leq 0$, from which it follows that the minimum of $\Delta(t)$ is reached at $t = r$ and amounts to $l_0(r) + (1+r)^{T-1} l_{T-1}(r) = v(r)$. To sum up, a curve satisfying properties (28)-(29)-(30) is the inverse $w^{-1}(r)$ of the wage curve of some adequately defined Sekt economy. ■

Theorem 4 *The wage curves associated with T-period Sekt economies have no specific properties with regard to wage curves associated with T-period economies.*

Proof. According to Lemma 5, conditions (28)-(29)-(30) coincide with those obtained for the v-curves of T-period economies as stated in Definition 1 and Theorem 2. ■

Theorem 4 is noteworthy because Sekt economies are a small subset of T-period economies, and therefore it was expected that their wage curves would have more properties.

Corollary 1 *The wage curve of a T-period economy coincides with the wage curve of a unique T-period Sekt economy.*

Proof. A wage curve of a T-period economy satisfies the necessary conditions (W_T). By Theorem 4, any curve which satisfies these conditions is the wage curve of some T-period Sekt economy. ■

The conclusion completes the one obtained in the previous Section. But Theorem 3 is local (a curve with adequate local properties coincides locally with the wage curve of some *T*-period economy) and the above corollary global (a curve with adequate properties is the wage curve of a T-period Sekt economy, with explicit formulas to characterise that economy). This does not mean, however, that Theorem 3 loses any interest: as mentioned above, the degrees of freedom in the choice of the intermediate functions $l_1(t), ..., l_{T-2}(t)$ can be used to meet other requirements.

Corollary 2 *A curve which satisfies conditions (28)-(29)-(30) for r varying in an interval I is the wage curve of a unique Sekt economy on that interval.*

Proof. Restrict the above proof of Lemma 6 to that interval. ■

These results allow us to check if some given function is a wage curve of a *T*-period economy. Let us return to Definition 2: Condition (i) is the Ricardian trade-off property. The intuitive content of condition (ii) is that the wage decreases at the rate smaller than the rate of increase of $(1+r)^{T-1}$. Condition (iii) sets a somewhat similar restriction on the derivative of the wage. For instance, it turns out that the curve $w(r) = \exp(-r)$ is not the wage curve of an Austrian economy because the wage and its derivative decrease too rapidly. On the contrary, a positive and decreasing curve expressed as the ratio of two polynomials is the wage curve of an Austrian economy for some great enough value of T.

6. Conclusion

The wage curves of Austrian models have been fully characterised by their quantitative properties, which express the trade-off between the wage and the rate of profit but also set some limits on the rate of decrease of the wage and its derivative. When an arbitrarily given curve meets these conditions, explicit formulas allow us to build an Austrian model sustaining that wage curve. Moreover, the existence of degrees of freedom in the choice of the parameters of the underlying Austrian model opens the possibility to take into account additional conditions or constraints. The normal parameterisation has shown to be a powerful tool to prove these properties and its use can be generalised to any family of models.

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Contact: Christian Bidard, University Paris Ouest, EconomiX, 200 avenue de la République, F-92001 Nanterre, France.

E-mail: `christian.bidard@u-paris10.fr`