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# Anomalous Switch Points

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**Abstract**

This article presents examples of models of rent with decreasing wage frontiers; with unique, square cost-minimizing techniques at a given rate of profits; but without the wage maximization property. Anomalous switch points are highlighted, where an anomalous switch point has properties that contrast with generic switch points in models of circulating capital. This article presents a numerical example of a switch point along a single wage curve, with no other wage curve intersecting at the switch point. Other numerical examples are of switch points in which only the scale, not the operated processes, vary between the cost-minimizing techniques at the switch point. A fake switch point is presented in which prices vary only for goods which are not commodities under the non-cost minimizing technique, also known as ghost commodities.

**Keywords:** Cambridge Capital Controversy; Fixed Capital; Rent.

**JEL Codes:** B51; C67; D24; D33; Q13.

## 1. Introduction[1]

General joint production can lack properties of models of single production. Bidard (1997) emphasizes four properties. Wage frontiers, under single production, are monotonically decreasing. The cost-minimizing technique is unique away from switch points; prices are unique at switch points. The cost-minimizing system is square; given the wage or the rate of profits, prices are determined. The cost-minimizing system maximizes the wage at a given rate of profits. D'Agata (1983), in models of intensive rent, presents numerical examples with increasing wage frontiers, with non-unique cost-minimizing square techniques, and without the wage maximization property. This article presents numerical examples of rent with decreasing wage curves, with unique cost-minimizing square techniques, and without the wage maximization property.

[1] This article was improved from comments by an anonymous reviewer. All errors and infelicities remain the responsibility of the author.

Anomalous switch points are highlighted in these numerical examples. An anomalous switch point has properties that contrast with generic switch points in models of circulating capital. Two examples are given of switch points at which the processes that are operated at a positive level do not vary between the techniques cost-minimizing before and after the switch points. In one of these examples, three wage curves intersect at switch points. Another case is given in which three wage curves intersect at a switch point, but the two techniques that are cost-minimizing before and after the switch point lie on the same wage curve. A switch point in a last example lies on the wage frontier without wage curves intersecting at the switch point. These examples arise in models of rent, with the last example also including fixed capital. Some properties of circulating capital models can fail in models of joint production, while downward-sloping wage frontiers and unique, square cost-minimizing techniques are retained.

Switch points are an aspect of the analysis of the choice of technique. A technique is a set of discrete production processes, each specified by inputs and outputs. These processes, for a feasible technique, can be operated at a level such that all commodity inputs are reproduced, leaving a net output to meet the given final demand. A wage curve is associated with each technique. In a model of single production, with inputs of labor and circulating capital alone, the wage curve for the cost-minimizing technique, at a given wage or rate of profits, lies on the outer envelope of all wage curves. The outer envelope is also known as the wage frontier. Wage curves and the wage frontier slope down, with the maximum wage being earned at a rate of profits of zero and the maximum rate of profits being obtained at a wage of zero. This analysis is an aspect of the indirect approach (Kurz and Salvadori 1995: 50, 147-148). The analysis of the choice of technique by the construction of the outer wage envelope is not generally applicable to models with joint production, including models with unproduced scarce natural resources for the use of whose services a rent is paid. The examples in this article have a unique cost-minimizing techniques away from switch points. Not all models with joint production have this property.

Generic switch points have various properties in models of circulating capital. The rate of profits at which they occur is independent of the numeraire. Two wage curves intersect at a switch point. The cost-minimizing technique changes. One technique is cost-minimizing for a rate of profits slightly less than at the switch point, and another is cost-minimizing at a slightly higher rate of profits. One process replaces another in an industry. The commodity output from that industry is produced under both techniques. Other processes can be introduced or removed if they produce commodities used only as inputs in one or the other process, as in the Samuelson-Garegnani model (Garegnani 1970).

The anomalous switch points presented here are not flukes. A switch point is a fluke when any perturbation of some parameters, such as coefficients of production, destroys defining features of the switch point. The concept of a fluke goes back to the 1966 symposium on capital theory in the *Quarterly Journal of Economics*:

> If, by a fluke more than one switch of technique happened to take place at exactly the same point, the nonzero columns [of the matrix formed by the difference of two Leontief matrices] would be more than one (Pasinetti 1966: 511).

Other participants recognize this fluke case in which four wage curves intersect at a switch point, with processes replacing one another in two industries:

> 'Adjacent' techniques on two sides of a switching point of a switching point will usually differ from each other only with respect to *one activity* (Bruno, Burmeister, Sheshinski 1966: 542).

Two wage curves tangent at a switch point is another fluke case. A perturbation leads to either the reswitching of techniques or of one cost-minimizing technique around the rate of profits at which the switch point formerly existed. Other fluke cases arise when a switch point exists at the maximum wage or the maximum rate of profits:

> Cases with multiple roots or cases in which the curves cross only at end points… These … are cases in which one technique can be ignored since it is dominated (Bruno, Burmeister, Sheshinski 1966: 534).

Fluke switch points exist in both models of single and joint production. Vienneau (2021) examines fluke switch points in pure fixed capital models, while Vienneau (2022) partitions a parameter space, with fluke switch points, in a model of extensive rent. Vienneau (2024) looks at fluke switch points in a model of non-competitive markets with single production. The characterization of fluke switch points is useful for analyzing structural economic dynamics with a choice of technique.

Bidard and Klimovsky (2004) describe an intersection of two wage curves with only one cost-minimizing technique as a fake switch point. Fake switch points can only occur in models of joint production. In their example, the rate of profits at which the fake switch point occurs depends on the numeraire. The prices of commodities produced by both techniques vary between the techniques, which is not possible at a genuine switch point. This article notes properties of selected fake switch points in the examples, including independence from the numeraire of the rate of profits at which one occurs. For some fakes, the only prices that vary between the techniques are for goods that are not commodities under one of the techniques.

The anomalous switch points presented in this article are from examples of models of rent. Extensive rent occurs when multiple types of land must be cultivated to satisfy requirements for use. Requirements for use are specified here by final demand, also known as required net output. Landlords obtain extensive rent on a type of land, except for flukes, when that type of land is farmed to the extent of its endowment, with only one process being operated on that type. Intensive rent occurs when more than one process must be operated on a single homogeneous type of land to satisfy requirements for use. With only one price prevailing for produced commodities and only one rate of profits being obtained in the system of prices of production, different amounts of rent per acre must be paid on the different types of land that are fully farmed. The analysis of the choice of technique shows, in models combining extensive and intensive rent, the conditions under which types of land would be specialized in producing specific commodities. Bidard (2004), Erreygers (1995), Kurz and Salvadori (1995), Quadrio Curzio (1980), Schefold (1989), and Woods (1990) develop models of rent in the post-Sraffian tradition.

The concept of a solving subsystem (Quadrio Curzio and Pellizzari 2010: 67-68) clarifies how a switch point can lie along a single wage curve. A system of equations for prices is associated with each technique. Each operated process contributes an equation equating revenues and costs. The revenues can include the prices of joint products, and costs include a charge for the rate of profits on advanced capital goods. A last equation specifies the value of the numeraire as unity. In models of extensive rent, a subsystem can be formed from the processes that characterize industrial processes, with no inputs from land, and processes run on lands that are not scarce. The resulting subsystem, with the equation for the numeraire concatenated, can be solved, given the rate of profits, for the wage and the prices of produced commodities. In models of intensive rent, the solving subsystem includes the equations for industrial processes and a linear combination of the equations for the processes that operate on one type of land to the limits of its endowment. As Sraffa (1960: 76) explains, a variable for rent is eliminated by this linear combination. In the case of extensive rent, with no joint production otherwise, the solving subsystem also applies to a model of single production. In any case, the solution to the solving subsystem can then be used to find rents. The example in Section 5 extends the concept of a solving subsystem to a case with extensive rent and fixed capital. I do not know if the concept of a solving subsystem can usefully apply to joint production more generally.

The center of a pure fixed capital system (Schefold 1989: 147-149; 152-159) helps solve the price system of a pure fixed capital system. Joint utilization of machines does not exist in any process in a model of pure fixed capital. Old machines are not consumer goods. In the example, a single commodity is a consumption good and acts as numeraire. Old machines may be freely disposed of; no cost arises in junking a machine, including before its technical life is complete. This assumption cannot be made in models appropriate for ecological economics. Nice properties of single production systems generalize to models of pure fixed capital. In particular, the "determination of the cost-minimizing technique is independent of the structure of requirements for use" (Huang, 2019). The cost-minimizing technique can be determined by the construction of the wage frontier as the outer envelope of wage curves. These properties are not retained in the combination of pure fixed capital with scarce land. The center still helps solve the price system.

This article is the first, as far as I know, to highlight properties of anomalous switch points, properties that cannot appear in models of circulating capital. Kurz and Salvadori (1995) present models of rent with multiple agricultural commodities, including numerical examples. A numerical example with fixed capital and extensive rent has not been previously presented. Non-fluke switch points can occur with no intersections of wage curves, with two intersecting wage curves, or with three intersecting curves. Bidard and Klimovsky (2004) argue that the latter is necessary in their example of a model of joint production. The techniques that are cost-minimizing can operate the same processes, at different scales. Furthermore, some variations in properties of selected fake switch points are noted.

The remainder of this article consists of five sections. The next section recalls the indirect approach for the analysis of the choice of technique and outlines the derivation of the wage frontier in that case. The third section presents a reswitching example in a model of extensive rents. Which processes are operated do not vary between cost-minimizing techniques at switch points in the example. The fourth section describes an example with multiple agricultural commodities and extensive and intensive rent. With one specification of final demand, switch points are also such that which processes are operated do not vary between cost-minimizing techniques. With another specification of final demand, a switch point appears in which the cost-minimizing techniques that vary lie along the same wage curve. The fifth section presents an example which combines fixed capital and rent. For one switch point in the example, the cost-minimizing techniques that vary also lie along the same wage curve. The number of wage curves that intersect at switch points vary, with some intersecting wage curves being for techniques that are not cost-minimizing. The final section concludes. An appendix justifies claims in the main text.

## 2. Wage Curves, the Wage Frontier, and Switch Points with Single Production

The cost-minimizing technique can be found by either the direct or the indirect approach. Since properties of switch points are examined in this article, this section presents an exposition of the indirect approach, in which wage curves are constructed for each technique. The indirect approach, as presented here, does not necessarily apply to models of joint production.

**Table 1: Parameters and Variables**

| Symbol | Type | Definition |
|---|---|---|
| $n$ | Parameter | Number of produced commodities, with $n > 0$. |
| $\mathbf{a}_0$ | Parameter | A $n$-element row vector; each element is the person-years of labor needed to operate a process at unit level. |
| $\mathbf{A}$ | Parameter | A $n$ x $n$ input matrix; each column specifies the physical inputs of produced commodities needed to operate a process at unit level. |
| $\mathbf{B}$ | Parameter | A $n$ x $n$ output matrix; each column is the physical outputs from operating a process at unit level. It is the identity matrix for single production. |
| $\mathbf{d}$ | Parameter | A $n$-element column vector representing requirements for use. Each element is the physical quantity of the corresponding commodity that must be in net output. Also, the numeraire. |
| $\mathbf{q}$ | Variable | A $n$-element column vector. The elements of the vector are the levels at which the processes are operated. |
| $\mathbf{p}$ | Variable | A $n$-element row vector of prices. |
| $r$ | Variable | The rate of profits. A pure number. |
| $w$ | Variable | The wage, in numeraire units per person-year. |

Coefficients of production are found for each technique in the indirect approach. The technology is represented by a finite number of techniques. A technique of production is specified by a row vector of direct labor coefficients, a square matrix of input coefficients, and a square matrix of output coefficients (Table 1).

In models of single production, the matrix of output coefficients is the identity matrix and each process in a technique is clearly operated in a single industry. The specification of the data for determining a long period position also includes the requirements for use. This vector is also known as required net output or final demand.

The coefficients of production for each technique are assumed to satisfy some conditions. The assumptions stated here are stronger than necessary. All coefficients of production and all components of final demand are assumed to be non-negative. Direct labor is needed to operate each process in a technique. In other words, each element of $\mathbf{a}_0$ is positive. All processes require some commodity input. Each column of **A** contains at least one positive element. All commodities are required as inputs, either directly or indirectly, to produce each commodity. The matrix **A** is indecomposable. The processes in the technique can be operated at some level such that a positive net output is produced. In other words, the eigenvalue of **A** maximum in magnitude is real, positive, and less than unity. Since only single production is considered in this section, the matrix **B** is an identity matrix.

For a technique to be cost-minimizing at a given rate of profits, it must at least be feasible and have associated non-negative prices of production. A technique is feasible in a model of single production if a vector $\mathbf{q}$ of levels of operation of processes exists such that equation 1 holds:

$$(\mathbf{B} - \mathbf{A}) \cdot \mathbf{q} = \mathbf{d} \tag{1}$$

The levels of operation must be non-negative:

$$\mathbf{q} \geq \mathbf{0} \tag{2}$$

Under the assumptions, all processes in the technique, with single production, are operated to satisfy requirements for use.

Prices of production satisfy the equality in equation 3:

$$\mathbf{p} \cdot \mathbf{A} \cdot (1 + r) + w \cdot \mathbf{a}_0 = \mathbf{p} \cdot \mathbf{B} \tag{3}$$

The price of the numeraire is unity:

$$\mathbf{p} \cdot \mathbf{d} = 1 \tag{4}$$

Prices of production are non-negative:

$$\mathbf{p} \geq \mathbf{0} \tag{5}$$

The solution of the quantity and price systems for models of single production have been extensively investigated.

The wage curve for a technique is found in solving the price system:

$$w(r) = \frac{1}{\mathbf{a}_0 \cdot [\mathbf{B} - (1 + r) \cdot \mathbf{A}]^{-1} \cdot \mathbf{d}} \tag{6}$$

The wage curves for all techniques can be graphed in the same space. The technique for which the wage is highest at a given rate of profits is cost-minimizing in the model of single production. The wage frontier is the outer envelope of all wage curves.

A switch point exists when two wage curves intersect on the frontier. A switch point in circulating capital models is at a rate of profits that does not depend on the numeraire. The cost-minimizing technique changes at switch point. Flukes aside, two wage curves intersect on the frontier. The two techniques with wage curves intersecting at a switch point differ in the process operated in one industry, where the commodity output from the industry is produced under both techniques. Other processes can be introduced or removed if they produce commodities used only as inputs in one or the other process, as in the Samuelson-Garegnani model (Garegnani 1970). The remainder of this article consists of the presentations of examples of models of rent where the analysis of the choice of technique and the resulting switch points differ from the single production case in some way.

## 3. A Reswitching Example with Extensive Rent

The technology for the first example consists of three processes, as shown in Table 2. Iron is produced with the first process, and corn with the other two processes. The two corn-producing processes require inputs of the services of a different type of land. Each process exhibits constant returns to scale, up to the limits imposed by the endowments of the types of land. These endowments consist of 100 acres of each type. The specification of the parameters of this numerical example is completed by defining required net output. Accordingly, suppose final demand consists of 125 bushels of corn. The final demand is also the numeraire.

**Table 2: Technology for an Example with Extensive Rent**

| Inputs | Process | | |
|---|---|---|---|
| | I | II | III |
| Labor (Person-Years) | $a_{0,1} = 1$ | $a_{0,2} = 9/10$ | $a_{0,3} = 3/5$ |
| Type-1 Land (Acres) | $c_{1,1} = 0$ | $c_{1,2} = 1$ | $c_{1,3} = 0$ |
| Type-2 Land (Acres) | $c_{2,1} = 0$ | $c_{2,2} = 0$ | $c_{2,3} = 49/50$ |
| Iron (Tons) | $a_{1,1} = 9/20$ | $a_{1,2} = 1/40$ | $a_{1,3} = 3/2000$ |
| Corn (Bushels) | $a_{2,1} = 2$ | $a_{2,2} = 1/10$ | $a_{2,3} = 9/20$ |
| OUTPUTS | 1 ton iron | 1 bushel corn | 1 bushel corn |

The processes constituting the technology can be combined in several ways to produce final demand. Table 3 shows possible techniques. The quantity flows and price systems for the Alpha and Beta techniques are as in the model of single production. Epsilon extends Alpha to produce corn with process III to the limit of the endowment of type-2 land. Theta extends Beta to produce corn to the limit of the endowment of type-1 land. Only Epsilon and Theta can feasibly produce the given final demand of 125 bushels (see A.1 in the appendix).

**Table 3: Techniques for an Example with Extensive Rent**

| Technique | Processes | Land | |
|---|---|---|---|
| | | Type 1 | Type 2 |
| Alpha | I, II | Partially Farmed | Fallow |
| Beta | I, III | Fallow | Partially Farmed |
| Epsilon | I, II, III | Partially Farmed | Fully Farmed |
| Theta | I, II, III | Fully Farmed | Partially Farmed |

The price equations for a technique in which both lands are farmed include rent on the scarce land. For example, consider the price equations for Epsilon:

$$\left(p_1^{\varepsilon} \cdot a_{1,1} + p_2^{\varepsilon} \cdot a_{2,1}\right) \cdot (1 + r) + w_{\varepsilon} \cdot a_{0,1} = p_1^{\varepsilon} \tag{7}$$

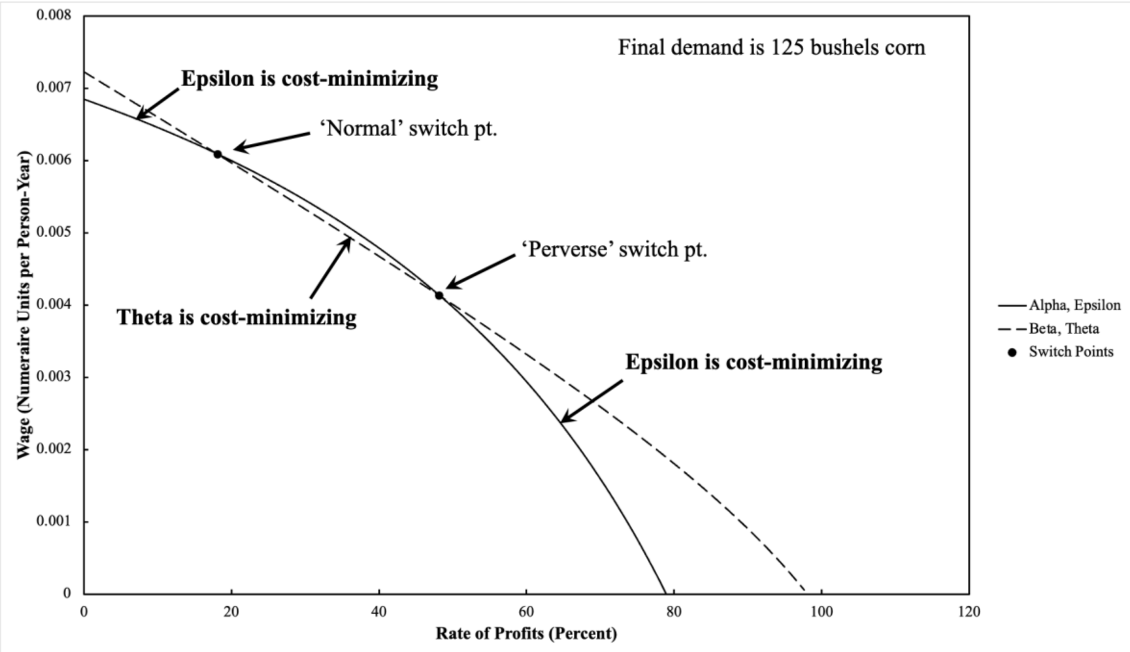
$$\left(p_1^{\varepsilon} \cdot a_{1,2} + p_2^{\varepsilon} \cdot a_{2,2}\right) \cdot (1 + r) + w_{\varepsilon} \cdot a_{0,2} = p_2^{\varepsilon} \tag{8}$$

$$\left(p_1^{\varepsilon} \cdot a_{1,3} + p_2^{\varepsilon} \cdot a_{2,3}\right) \cdot (1 + r) + \varrho_2^{\varepsilon} \cdot c_{2,3} + w_{\varepsilon} \cdot a_{0,3} = p_2^{\varepsilon} \tag{9}$$

$$p_1^{\varepsilon} \cdot d_1 + p_2^{\varepsilon} \cdot d_2 = 1 \tag{10}$$

Rent on type-1 land does not appear in Equation (8), since type-1 land is free under Epsilon. Equations (7) and (8), along with equation (10), which specifies the numeraire, can be solved to yield the wage and the prices of the two produced commodities as functions of the rate of profits. Equations and (7), (8), and (10) constitute the solving subsystem for Epsilon. The solving subsystem and the resulting wage curve, as seen in Figure 1, also apply to Alpha. The wage curves in models of extensive rent accordingly have all the properties of wage curves in models of circulating capital.

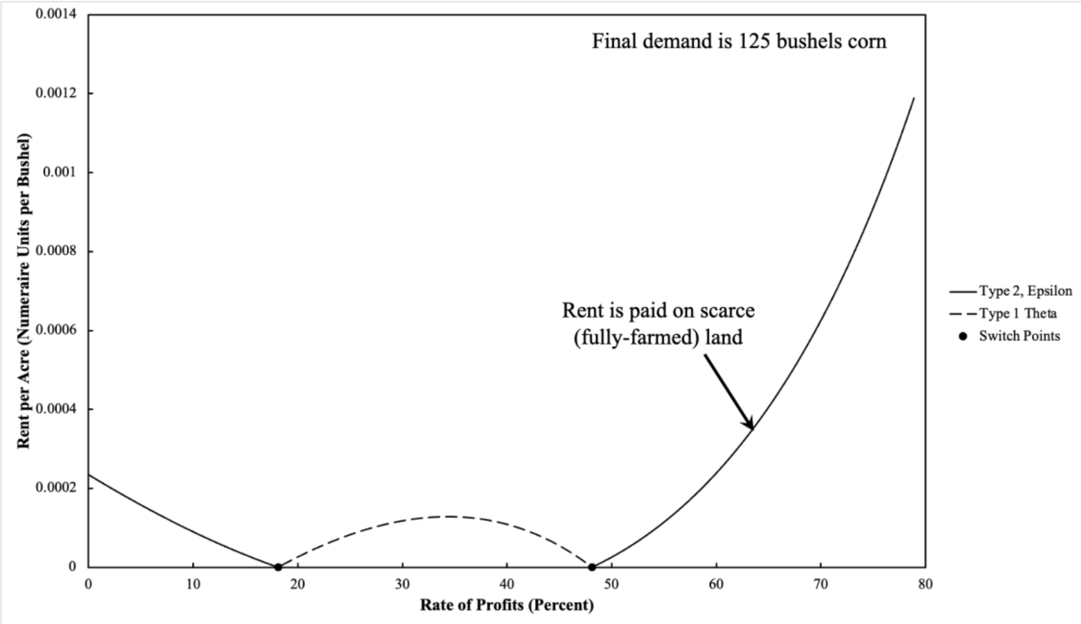
**Figure 1: The Cost-Maximizing Technique for the Example with Extensive Rent**




The solving subsystem for Theta applies to Beta. Thus, the wage curves for Beta and Theta are the same. Once the wage and the prices of iron and corn have been solved for

Epsilon, Equation (9) yields the rent curve for Epsilon. The rent on type-1 land under Theta is found similarly. Figure 2 graphs the rent curves for this numerical example.

**Figure 2: Rent Curves for the Example with Extensive Rent**




The cost-minimizing technique is feasible; it does not pay a negative rent; and it is such that extra profits cannot be obtained, at prices associated with the technique, by operating processes outside the technique. In this example, the wage frontier is the inner envelope of the wage curves, as shown in Figure 1. This is an example of the reswitching of techniques, with the second switch point being perverse in the sense that it does not conform to obsolete marginalist intuition. Since the first switch point is not perverse, I call it normal. The perverse switch point exhibits capital-reversing. Around this switch point, a higher wage or lower rate of profits is associated with firms ultimately adopting a technique in which more labor is hired to produce the given net output. The more labor-intensive technique is also less capital-intensive. Prices of production are not indices of relative scarcity.

The switch points between Epsilon and Theta are at intersections of two wage curves, as in models of single production. The switch points, however, are not a matter of one process replacing another. They are anomalous switch points in that the same three processes are operated under both techniques. They differ in the scale of operations for these processes. Under Theta, type-1 land is scarce, and under Epsilon, type-2 land is scarce. Landlords obtain extensive rent under both techniques.

## 4. A Reswitching Example with Multiple Agricultural Commodities

For another example of anomalous switch points, consider an economy in which more than one agricultural commodity can be grown on each of two types of land. Table 4 specifies the technology. As in the previous example, iron is an industrial commodity produced without the use of services of land. Two separate processes are known for harvesting wheat on either of two types of land. Another pair of processes is known for rye. An endowment of 100 acres of each type of land is available.

**Table 4: Technology for an Example with Multiple Agricultural Commodities**

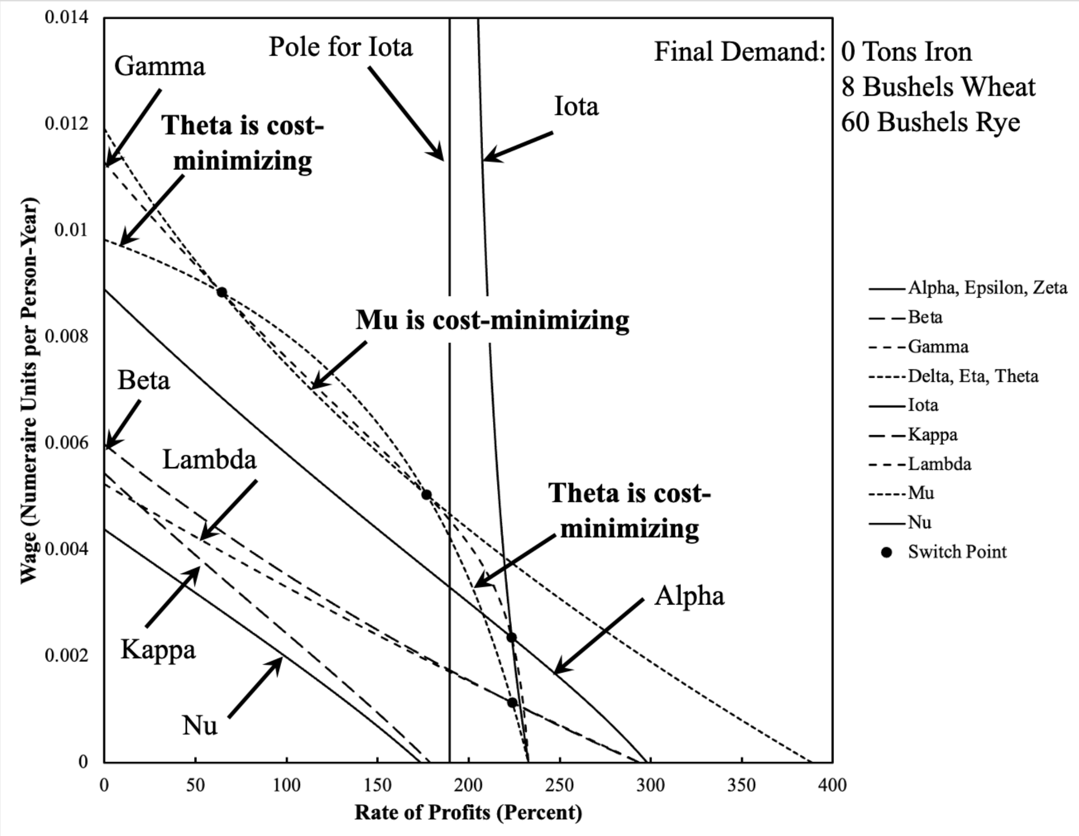
| Inputs | Industry | | | | |
|---|---|---|---|---|---|
| | Iron | Wheat | | Rye | |
| | I | II | III | IV | V |
| Labor (Person-Yrs) | $a_{0,1} = \frac{1}{3}$ | $a_{0,2} = \frac{5}{2}$ | $a_{0,3} = \frac{7}{20}$ | $a_{0,4}=1$ | $a_{0,5} = \frac{3}{2}$ |
| Type-1 Land (Acres) | $c_{1,1} = 0$ | $c_{1.2} = 1$ | $c_{1,3} = 0$ | $c_{1,4}=2$ | $c_{1,5} = 0$ |
| Type-2 Land (Acres) | $c_{2,1} = 0$ | $c_{2,2} = 0$ | $c_{2.3} = 5$ | $c_{2,4} = 0$ | $c_{2,5}=1$ |
| Iron (Tons) | $a_{1,1} = \frac{1}{6}$ | $a_{1,2} = \frac{1}{200}$ | $a_{1,3} = \frac{1}{300}$ | $a_{1,4} = 1$ | $a_{1,5}=0$ |
| Wheat (Bushels) | $a_{2,1} = \frac{1}{200}$ | $a_{2,2} = \frac{1}{4}$ | $a_{2,3} = \frac{3}{10}$ | $a_{2,4} = 0$ | $a_{2,5} = \frac{1}{4}$ |
| Rye (Bushels) | $a_{3,1} = \frac{1}{300}$ | $a_{3,2} = \frac{1}{300}$ | $a_{3,3} = 0$ | $a_{3,4} = 0$ | $a_{3,5} = 0$ |
| **Output** | 1 Ton Iron | 1 Bushel Wheat | 1 Bushel Wheat | 1 Bushel Rye | 1 Bushel Rye |

**Table 5: Techniques for an Example with Multiple Agricultural Commodities**

| Technique | Processes | Type-1 Land | Type-2 Land |
|---|---|---|---|
| Alpha | I, II, IV | Partially Farmed | Fallow |
| Beta | I, II, V | Partially Farmed | Partially Farmed |
| Gamma | I, III, IV | Partially Farmed | Partially Farmed |
| Delta | I, III, V | Fallow | Partially Farmed |
| Epsilon | I, II, III, IV | Partially Farmed | Fully Farmed |
| Zeta | I, II, IV, V | Partially Farmed | Fully Farmed |
| Eta | I, II, III, V | Fully Farmed | Partially Farmed |
| Theta | I, III, IV, V | Fully Farmed | Partially Farmed |
| Iota | I, II, III, IV | Fully Farmed | Partially Farmed |
| Kappa | I, II, IV, V | Fully Farmed | Partially Farmed |
| Lambda | I, II, III, V | Partially Farmed | Fully Farmed |
| Mu | I, III, IV, V | Partially Farmed | Fully Farmed |
| Nu | I, II, III, IV, V | Fully Farmed | Fully Farmed |

The capitalists choose from among 13 techniques (Table 5). Land is not scarce for techniques Alpha, Beta, Gamma, and Delta. Only one type of land is (partially) farmed in Alpha and Delta. The lands are specialized in Beta and Gamma. Landlords of one type of land obtain extensive rent in Epsilon, Zeta, Eta, and Theta. Epsilon and Zeta operate the same processes on non-scarce type-1 land as Alpha. Their solving subsystems are the same as Alpha's systems of equations for prices. Likewise, the Eta and Theta techniques have the same solving subsystem as Delta. The Iota, Kappa, Lambda, and Mu techniques operate the same processes as Epsilon, Zeta, Eta, and Theta, respectively. They differ in that landlords obtain intensive under them on the other type of land than is scarce in the corresponding technique with extensive rent. Under Nu, landlords obtain intensive rent on both types of land.

**Figure 3: Wage Curves for the Example with Multiple Agricultural Commodities**

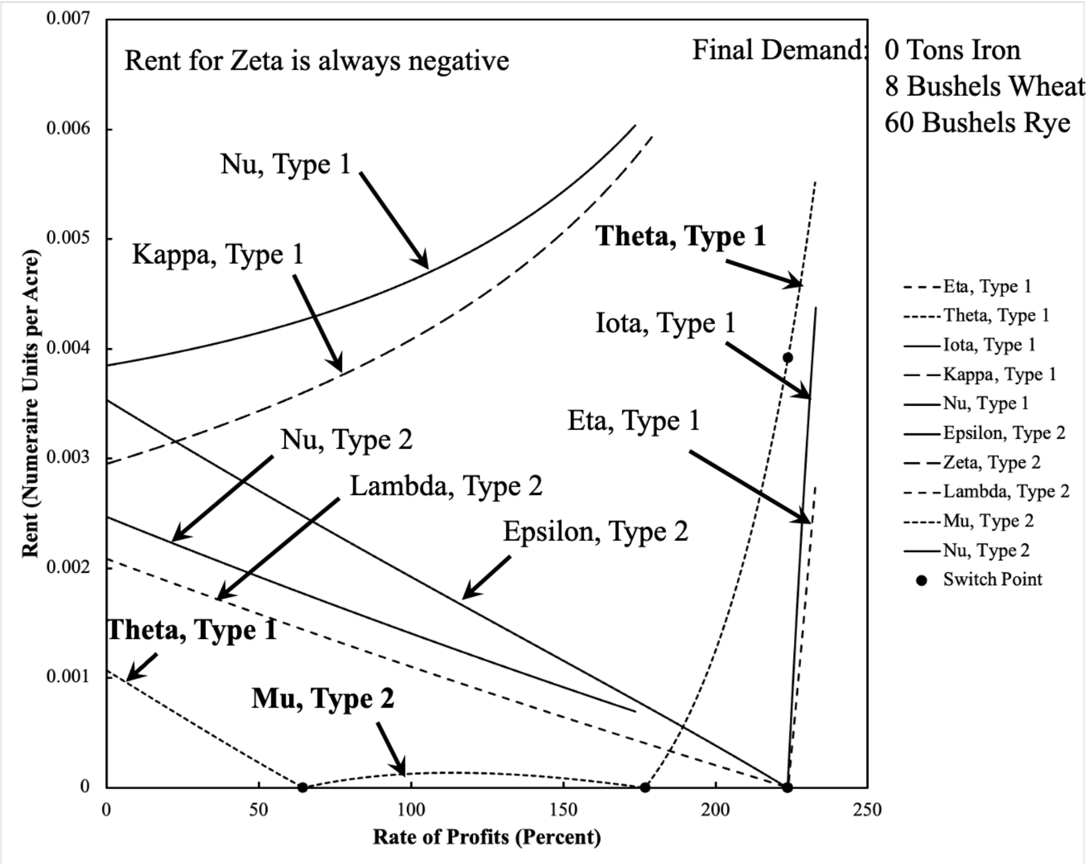



*4.1 The Example Heavy on the Rye*

Suppose required net output consists of 8 bushels of wheat and 60 bushels of rye. With this final demand, Beta, Theta, Kappa, Lambda, and Mu are feasible (see A.2 in the appendix). The wage curves, with required net output as numeraire, are plotted in Figure 3. Figure 4 shows the rent curves. Some intersections of wage curves are not labeled switch points. For instance, consider the intersection between the wage curves for Kappa and Lambda. Under Kappa, rye is grown on type-1 land, but under Lambda wheat is grown on type-2 land. Along with the change in which land is scarce, this is a difference for processes in two industries, wheat and rye. The wage curve for Iota stands out. The rate of profits at which the asymptote occurs varies with the numeraire. Rent on type-1 land at Iota prices is only positive after the intersection with the wage curves for Alpha and Gamma, which are not feasible either.

This example, with this specification of final demand, is another reswitching example, here between Theta and Mu. The switch points between Theta and Mu are anomalous in that the same four processes are operated under both techniques. Landlords obtain extensive rent under Theta and intensive rent under Mu. They are also anomalous in that they are intersections of three wage curves, not two.

**Figure 4: Rent Curves for the Example with Multiple Agricultural Commodities**




The intersection between the Beta, Theta, and Lambda wage curves is a fake switch point. Prices of iron, wheat, and corn do not vary among Beta, Delta, and Lambda, as at a genuine switch point. Beta pays extra profits of zero, at Beta prices, on the process for growing wheat on type-2 land (see appendix). This condition of zero extra profits would obtain at a genuine switch point with Beta and techniques that operate process III, like Theta and Lambda. But Beta pays extra profits on the process for growing rye on type-1 land, for any rate of profits. Thus, Beta cannot be cost-minimizing before, after, or at this fake switch point. Rent on type-2 land is zero, at Lambda prices, and negative for rates of profits greater than at the fake switch point. But Lambda also pays extra profits on growing rye on type-1 land, for any rate of profits for which rent is non-negative. Thus, this cannot be a switch point for Lambda. Finally, Theta pays a positive rent on type-1 land at this fake switch point, which would not be the case if this were a genuine switch point.

This example demonstrates that the rate of profits at which a fake switch point occurs need not depend on the specification of the numeraire, differently from what is the case in the example from Bidard and Klimovsky (2004). Furthermore, the cost-minimizing technique does not maximize the wage, at rates of profits greater than at the fake, even if only feasible techniques are considered.

*4.2 The Example with an Even Balance of Wheat and Rye*

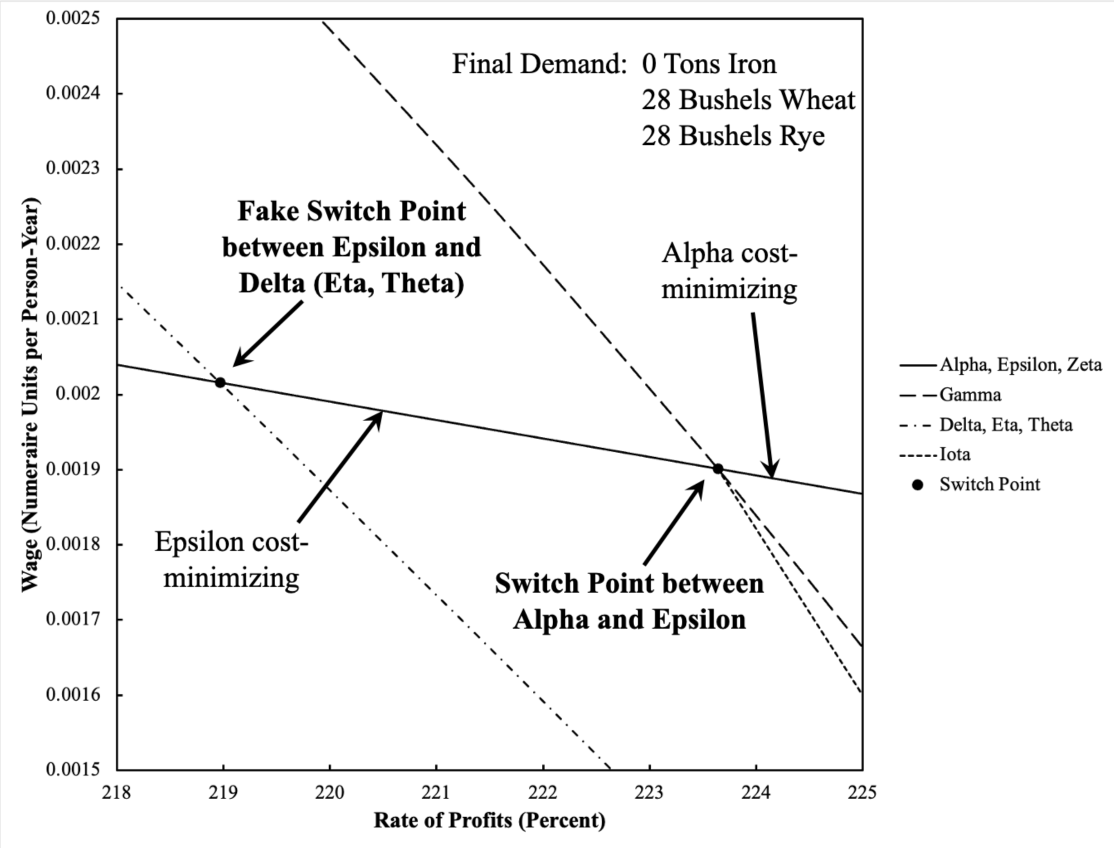
For another example, assume that that final demand consists of 28 bushels of wheat and 28 bushels of rye. Technology and endowments of land remain as in Section 4.1. With this new specification of final demand, Alpha, Beta, Epsilon and Lambda are feasible. Figure 5 shows a portion of the wage curves, around a selected fake and a genuine switch point.

Epsilon is cost-minimizing for rates of profits up to the switch point, and Alpha is cost-minimizing thereafter (see appendix). This is an example of the reswitching of the order of efficiency (Vienneau 2022) in the range of the rate of profits where Epsilon is cost-minimizing. Epsilon differs from Alpha in that wheat is grown on type-2 land under Epsilon to the extent of its endowment. Type-2 land is not cultivated under Alpha. Under Epsilon, rent on type-2 land is positive before the switch point, zero at the switch point, and negative thereafter. The solving subsystem for Epsilon is the price system for Alpha. Accordingly, the switch point for Alpha and Epsilon lies on a single wage curve, an anomalous phenomenon not to be seen in models of circulating capital alone.

Additional wage curves intersect here, which is another anomalous aspect of the switch point. Suppose final demand were low enough to make only Alpha, Beta, Gamma, and Delta were feasible. For example, this would be a circulating capital model, without scarce land, if final demand consisted of 10 bushels of wheat and 10 bushels of rye. Then the switch point would be between Gamma and Alpha, lying on the outer wage frontier, constructed from the outer envelope of the four wage curves for feasible techniques. Intuition trained on models of circulating capital is confirmed. On the other hand, suppose final demand consisted of 30 bushels of wheat and 30 bushels of rye. Beta, Epsilon, Iota, Kappa, and Lambda would be feasible. The switch point would become one between Epsilon and Iota. Epsilon and Iota operate the same processes, but at different scales. The rent on type-1 land is zero at the switch point under Iota, and positive for a larger rate of profits less than the maximum. Firms are indifferent, at the switch point, between growing wheat on type-1 or type-2 land under Alpha and Gamma. They grow wheat on both lands under Epsilon and Iota, but which is scarce varies.

Figure 5 also illustrates a fake switch point. The wage curves for Delta and Epsilon intersect at the fake switch point. The wage curve for Delta is also the wage curve for both Eta and Theta. Epsilon is the unique cost-minimizing technique at and around this fake switch point. The prices of produced commodities (iron, wheat, and rye) differ, at the switch point, between the techniques for the two intersecting wage curves. In this sense, the fake switch point resembles the one in the example from Bidard and Klimovsky. The rent on type-2 land is positive under Epsilon, which would not be the case if this switch point were genuine. Nor are the rents on type-1 land zero under Eta and Theta at this fake.

**Figure 5: Enlarged Wage Curves for the Example with Multiple Agricultural Commodities**




As with the example in Section 3, this example illustrates that which processes are operated in cost-minimizing techniques need not vary at a switch point. It also illustrates a switch point that lies along a single wage curve. Additional wage curves, for non-feasible techniques, intersect at the switch points. The example also illustrates fake switch points. The rate of profits at which the fake occurs does not depend on the numeraire for the example in Section 4.1. The rate of profits does depend on the numeraire for the fake in this section.

## 5. An Example with Fixed Capital and Rent

A final example illustrates a switch point along a single wage curve, with no intersecting wage curves at the switch point. A fake switch point illustrates a case in which prices vary at the switch point only for ghost commodities under the non-cost-minimizing technique. A commodity is a ghost commodity if it is not produced or is not a commodity for the technique under consideration (Bidard 2016). A type of land is not a commodity if it is free because it is not fully farmed. A type of an old machine is a ghost commodity if it is not produced or it is discarded unused under the technique. Ghost commodities must be taken into account when analyzing the choice of technique.

This anomalous switch and fake switch point arises in an example that combines scarce land and fixed capital. Fixed capital is manifested by a produced machine that can last several years in production. Land and fixed capital are the only aspects of joint production that appear in the example. The analysis of the choice of technique considers the economic life of the machine on each type of land. Free disposal is assumed. Old machines, whether run to the length of their physical life or not, can be discarded without cost. Old machines cannot be transferred from use on one type of land to another. This assumption is taken over from models of pure fixed capital.

**Table 6: Inputs for Processes for Example with Fixed Capital**

| Inputs | Processes | | | | | | |
|---|---|---|---|---|---|---|---|
| | **I** | **II** | **III** | **IV** | **V** | **VI** | **VII** |
| Labor | $a_{0,1} = 0.4$ | $a_{0,2} = 0.2$ | $a_{0,3} = 0.6$ | $a_{0,4} = 0.4$ | $a_{0,5} = 0.23$ | $a_{0,6} = 0.59$ | $a_{0,7} = 0.39$ |
| Type-1 Land | $c_{1,1} = 0$ | $c_{1,2} = 1$ | $c_{1,3} = 1$ | $c_{1,4} = 1$ | $c_{1,5} = 0$ | $c_{1,6} = 0$ | $c_{1,7} = 0$ |
| Type-2 Land | $c_{2,1} = 0$ | $c_{2,2} = 0$ | $c_{2,3} = 0$ | $c_{2,4} = 0$ | $c_{2,5} = 1$ | $c_{2,6} = 1$ | $c_{2,7} = 1$ |
| Corn | $a_{1,1} = 0.1$ | $a_{1,2} = 0.4$ | $a_{1,3} = 0.578$ | $a_{1,4} = 0.6$ | $a_{1,5} = 0.39$ | $a_{1,6} = 0.59$ | $a_{1,7} = 0.61$ |
| New Machines | $a_{2,1} = 0$ | $a_{2,2} = 1$ | $a_{2,3} = 0$ | $a_{2,4} = 0$ | $a_{2,5} = 1$ | $a_{2,6} = 0$ | $a_{2,7} = 0$ |
| Type-1 1-Yr. Old | $a_{3,1} = 0$ | $a_{3,2} = 0$ | $a_{3,3} = 1$ | $a_{3,4} = 0$ | $a_{3,5} = 0$ | $a_{3,6} = 0$ | $a_{3,7} = 0$ |
| Type-1 2-Yr. Old | $a_{4,1} = 0$ | $a_{4,2} = 0$ | $a_{4,3} = 0$ | $a_{4,4} = 1$ | $a_{4,5} = 0$ | $a_{4,6} = 0$ | $a_{4,7} = 0$ |
| Type-2 1-Yr. Old | $a_{5,1} = 0$ | $a_{5,2} = 0$ | $a_{5,3} = 0$ | $a_{5,4} = 0$ | $a_{5,5} = 0$ | $a_{5,6} = 1$ | $a_{5,7} = 0$ |
| Type-2 2-Yr. Old | $a_{6,1} = 0$ | $a_{6,2} = 0$ | $a_{6,3} = 0$ | $a_{6,4} = 0$ | $a_{6,5} = 0$ | $a_{6,6} = 0$ | $a_{6,7} = 1$ |

Tables 6 and 7 specify the technology. This technology extends an example of fixed capital from Baldone (1974). Labor uses circulating capital to manufacture a machine in process I. The machine has a physical life of three years. Labor uses circulating capital and the machine to produce corn on type-1 land in processes II, III, and IV. The machine is operated on type-2 land in processes V, VI, and VII. A process that produces corn jointly produces a machine one year older than the machine used as input, up to its physical life. One hundred acres of each type of land are assumed to exist. Final demand is for 87 bushels of corn, a level that ensures one or the other type of land is scarce. The numeraire is a bushel of corn.

Table 8 specifies the techniques that may be chosen with this technology. Alpha, Beta, and Gamma differ in the economic life of the machine on non-scarce, type-1 land. No processes are operated on type-2 land. Under Delta, Epsilon, and Zeta, on the other hand, type-1 land is not farmed at all, and the economic life of the machine varies among the techniques in the processes operated on type-2 land. The remaining techniques fully cultivate one or the other type of land and require rent to be paid to landlords.

**Table 7: Outputs for Processes for Example with Fixed Capital**

| Outputs | Processes | | | | | | |
|---|---|---|---|---|---|---|---|
| | **I** | **II** | **III** | **IV** | **V** | **VI** | **VII** |
| Corn | $b_{1,1} = 0$ | $b_{1,2} = 1$ | $b_{1,3} = 1$ | $b_{1,4} = 1$ | $b_{1,5} = 1$ | $b_{1,6} = 1$ | $b_{1,7} = 1$ |
| New Machines | $b_{2,1} = 1$ | $b_{2,2} = 0$ | $b_{2,3} = 0$ | $b_{2,4} = 0$ | $b_{2,5} = 0$ | $b_{2,6} = 0$ | $b_{2,7} = 0$ |
| Type-1 1-Yr. Old | $b_{3,1} = 0$ | $b_{3,2} = 1$ | $b_{3,3} = 0$ | $b_{3,4} = 0$ | $b_{3,5} = 0$ | $b_{3,6} = 0$ | $b_{3,7} = 0$ |
| Type-1 2-Yr. Old | $b_{4,1} = 0$ | $b_{4,2} = 0$ | $b_{4,3} = 1$ | $b_{4,4} = 0$ | $b_{4,5} = 0$ | $b_{4,6} = 0$ | $b_{4,7} = 0$ |
| Type-2 1-Yr. Old | $b_{5,1} = 0$ | $b_{5,2} = 0$ | $b_{5,3} = 0$ | $b_{5,4} = 0$ | $b_{5,5} = 1$ | $b_{5,6} = 0$ | $b_{5,7} = 0$ |
| Type-2 2-Yr. Old | $b_{6,1} = 0$ | $b_{6,2} = 0$ | $b_{6,3} = 0$ | $b_{6,4} = 0$ | $b_{6,5} = 0$ | $b_{6,6} = 1$ | $b_{6,7} = 0$ |

**Table 8: Techniques for Example with Fixed Capital**

| Technique | Processes | Type-1 Land | Type-2 Land |
|---|---|---|---|
| Alpha | I, II | Partially Farmed | Fallow |
| Beta | I, II, III | Partially Farmed | Fallow |
| Gamma | I, II, III, IV | Partially Farmed | Fallow |
| Delta | I, V | Fallow | Partially Farmed |
| Epsilon | I, V, VI | Fallow | Partially Farmed |
| Zeta | I, V, VI, VII | Fallow | Partially Farmed |
| Eta | I, II, V | Fully Farmed | Partially Farmed |
| Theta | I, II, III, V | Fully Farmed | Partially Farmed |
| Iota | I, II, III, IV, V | Fully Farmed | Partially Farmed |
| Kappa | I, II, V, VI | Fully Farmed | Partially Farmed |
| Lambda | I, II, III, V, VI | Fully Farmed | Partially Farmed |
| Mu | I, II, III, IV, V, VI | Fully Farmed | Partially Farmed |
| Nu | I, II, V, VI, VII | Fully Farmed | Partially Farmed |
| Xi | I, II, III, V, VI, VII | Fully Farmed | Partially Farmed |
| Omicron | I, II, III, IV, V, VI, VII | Fully Farmed | Partially Farmed |
| Pi | I, II, V | Partially Farmed | Fully Farmed |
| Rho | I, II, III, V | Partially Farmed | Fully Farmed |
| Sigma | I, II, III, IV, V | Partially Farmed | Fully Farmed |
| Tau | I, II, V, VI | Partially Farmed | Fully Farmed |
| Upsilon | I, II, III, V, VI | Partially Farmed | Fully Farmed |
| Phi | I, II, III, IV, V, VI | Partially Farmed | Fully Farmed |
| Chi | I, II, V, VI, VII | Partially Farmed | Fully Farmed |
| Psi | I, II, III, V, VI, VII | Partially Farmed | Fully Farmed |
| Omega | I, II, III, IV, V, VI, VII | Partially Farmed | Fully Farmed |

Under techniques Eta through Omicron, type-1 land is fully farmed and pays rent. Under Eta, Theta, and Iota, the machine is operated for only one year on type-2 land and then discarded. The techniques differ on the economic life of the machine on type-1 land. Under Kappa, Lambda, and Mu, the machine is operated for two years on type-2 land, while it is operated for its full physical life of three years under Nu, Xi, and Omicron. Under Pi through Omega, type-2 land is scarce and pays rent. Each technique between Eta and Omicron corresponds to a technique between Pi and Omega in which the same processes are operated. The economic life of the two types of machines are the same in these corresponding techniques. The scale at which the processes are run varies so as to vary which type of land is fully farmed.

*5.1 The Centre of the Solving Subsystem*

I consider the price equations for Omicron to illustrate the concepts of the solving subsystem and of the center. All seven processes are operated under Omicron, and type-1 land is scarce. Equations (11) through (17), in obvious notation, specify the price system for Omicron:

$$a_{1,1} \cdot (1 + r) + w_o \cdot a_{0,1} = p_0^o \quad (11)$$

$$\left(a_{1,2} + p_0^o\right) \cdot (1 + r) + \varrho_1^o \cdot c_{1,2} + w_o \cdot a_{0,2} = b_{1,2} + p_{1,1}^o \quad (12)$$

$$\left(a_{1,3} + p_{1,1}^o\right) \cdot (1 + r) + \varrho_1^o \cdot c_{1,3} + w_o \cdot a_{0,3} = b_{1,3} + p_{1,2}^o \quad (13)$$

$$\left(a_{1,4} + p_{1,2}^o\right) \cdot (1 + r) + \varrho_1^o \cdot c_{1,4} + w_o \cdot a_{0,4} = b_{1,4} \quad (14)$$

$$\left(a_{1,5} + p_0^o\right) \cdot (1 + r) + w_o \cdot a_{0,5} = b_{1,5} + p_{2,1}^o \quad (15)$$

$$\left(a_{1,6} + p_{2,1}^o\right) \cdot (1 + r) + w_o \cdot a_{0,6} = b_{1,6} + p_{2,2}^o \quad (16)$$

$$\left(a_{1,7} + p_{2,2}^o\right) \cdot (1 + r) + w_o \cdot a_{0,7} = b_{1,7} \quad (17)$$

Revenues for operating each process at a unit level are shown on the right-hand side of these equations. Revenues for the first process are obtained by selling new machines. Revenues for the second process result from products of both corn and a type-1 one-year old machine. That type-1 machine, in turn, enters into the advanced costs of the third process, and so on. Type-1 land obtains a rent, and type-2 land is free.

Equations (11), (15), (16), and (17) constitute the solving subsystem for Omicron. Given the rate of profits, the solving subsystem specifies the wage, the price of a new machine, and the prices of one-year old and two-year old machines when operated on free type-2 land. Equations (12), (13), and (14) can then be used to find the rent on type-1 land and the prices of one-year old and two-year old machines when operated on type-1 land. The solving subsystem for Omicron is also the solving subsystem for Zeta, Nu, and Xi. In all these techniques, the machine is run for its full physical life of three years on free type-2 land.

The prices of old type-2 machines can be eliminated from the solving subsystem for Omicron. Multiply both sides of Equation 15 by $(1 + r)^2$ to get Equation (18):

$$\left(a_{1,5} + p_0^o\right) \cdot (1 + r)^3 + w_o \cdot a_{0,5} \cdot (1 + r)^2 = b_{1,5} \cdot (1 + r)^2 + p_{2,1}^o \cdot (1 + r)^2 \quad (18)$$

Multiply both sides of Equation (16) by $(1 + r)$:

$$\left(a_{1,6} + p_{2,1}^o\right) \cdot (1+r)^2 + w_o \cdot a_{0,6} \cdot (1+r) = b_{1,6} \cdot (1+r) + p_{2,2}^o \cdot (1+r) \tag{19}$$

Sum Equations (18), (19), and (17) to yield Equation (20):

$$\begin{aligned} &p_0^o \cdot (1+r)^3 + a_{1,5} \cdot (1+r)^3 + a_{1,6} \cdot (1+r)^2 + a_{1,7} \cdot (1+r) \\ &+ w_o \cdot \left[a_{0,5} \cdot (1+r)^2 + a_{0,6} \cdot (1+r) + a_{0,7}\right] \\ &= b_{1,5} \cdot (1+r)^2 + b_{1,6} \cdot (1+r) + b_{1,7} \end{aligned} \tag{20}$$

Or:

$$\begin{aligned} &[p_0^o \quad 1] \cdot \begin{bmatrix} 0 & \dfrac{(1+r)^2}{b_{1,5} \cdot (1+r)^2 + b_{1,6} \cdot (1+r) + b_{1,7}} \\ a_{1,1} & \dfrac{a_{1,5} \cdot (1+r)^2 + a_{1,6} \cdot (1+r) + a_{1,7}}{b_{1,5} \cdot (1+r)^2 + b_{1,6} \cdot (1+r) + b_{1,7}} \end{bmatrix} \cdot (1+r) \\ &+ w_o \cdot \left[a_{0,1} \quad \frac{a_{0,5} \cdot (1+r)^2 + a_{0,6} \cdot (1+r) + a_{0,7}}{b_{1,5} \cdot (1+r)^2 + b_{1,6} \cdot (1+r) + b_{1,7}}\right] = [p_0^o \quad 1] \end{aligned} \tag{21}$$

With appropriate definitions of matrices, Equation (21) is rewritten as Equation (22):

$$[p_0^o \quad 1] \cdot \hat{\mathbf{A}}(r) \cdot (1+r) + w_o \cdot \hat{\mathbf{a}}_0(r) = [p_0^o \quad 1] \tag{22}$$

The ordered pair $\left(\hat{\mathbf{A}}(r), \hat{\mathbf{a}}_0(r)\right)$ is the center (Schefold 1989: 147-149; 152-159) for the solving subsystem for Omicron. Given the rate of profits, the system of equations (22) can be solved for the wage and the price of a new machine. Equation (22) has the form of the price system for a circulating capital model, with the exception of the dependence of the Leontief input matrix and the vector of direct labor coefficients on the rate of profits. Unlike in the model of circulating capital, the wage curve derived from the centre of a pure fixed capital system can slope up for part of its range. The wage frontier of a pure fixed capital system, however, decreases throughout its length (Baldone 1974, Varri 1974).

The prices of old type-1 machines can be similarly eliminated from the full price system for Omicron.

### *5.2 Cost-Minimization and Selected Switch Points*

The Omicron technique illustrates aspects of the analysis of the choice technique in models of fixed capital. Since all processes in the technology are operating under Omicron, no process obtains extra profits under prices of production for Omicron. It would be mistaken, however, to conclude that Omicron is cost-minimizing whenever the wage is positive in the price system for Omicron. The lower bound for which Omicron is cost-minimizing is set by the lowest rate of profits at which the price of a type-1 one-year old machine becomes non-negative. The upper bound is set by the price of a type-2 one-year old machine turning negative (see A.4 in the appendix). Within this range, the wage, the prices of machines of any type or vintage, and the rent on type-1 land are positive in the solution of the price system.

More generally, a technique in a model with fixed capital and non-transferrable old machines is cost-minimizing away from a switch point if and only if:

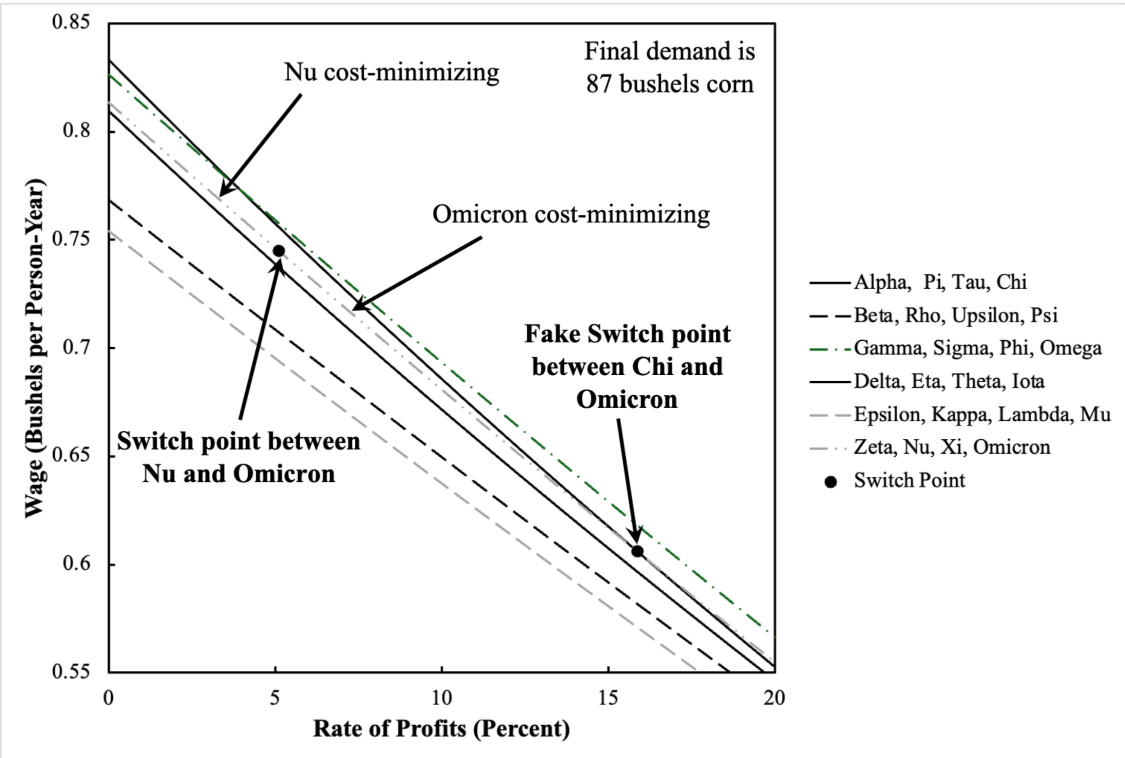
- The technique is feasible,

- The wage is non-negative,
- Rent on scarce land is positive,
- The prices of produced commodities (including new machines and whichever old machines are produced) are positive, and
- If the economic life of a machine is extended, a previously discarded older machine has a negative price, for each extension.

The last condition includes the comparison of techniques in which different types of land are scarce. This last condition also includes a comparison of techniques in which the economic life of a machine is extended to any extent, not just one year. For example, an analysis of the range of the rate of profits at which Pi is cost-minimizing includes examining the price systems for Theta, Rho, Iota, Sigma, Kappa, Tau, Nu, and Chi. The price of one vintage of an old machine is zero at a non-fluke switch point in which the economic life of a machine varies. The rent on both types of land is zero at a non-fluke switch point in which the type of land that is scarce varies.

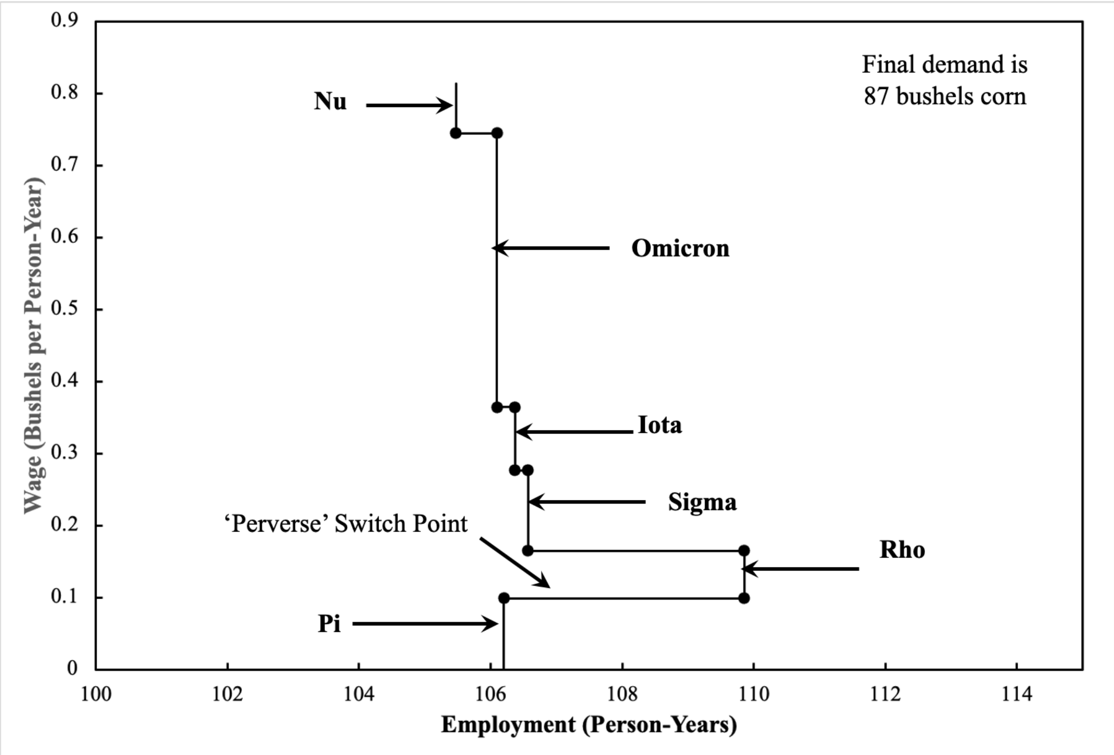
The analysis of the choice of the economic life of a machine allows for a discussion for the switch point and fake switch point in Figure 6. I found the switch point between Nu and Omicron to be the most surprising among these examples. The switch point lies along a single wage curve, and no other wage curve intersects that wage curve at the switch point. The economic life of a type-1 machine is extended from one to three years in a switch from Nu to Omicron. Since type-1 land pays a rent under both techniques, Nu and Omicron have the same solving system and thus the same wage curve.

**Figure 6: Enlarged Wage Curves for an Example with Fixed Capital and Rent**



Prices vary at the selected fake switch point only for ghost commodities (Bidard 2016) under the non-cost-minimizing technique, that is, Chi. Type-1 rent is positive for Omicron and non-scarce for Chi. The price of a type-1 one-year old machine is positive under Omicron and not produced under Chi. The prices of corn, new machines, type-2 one-year old machines, and type-2 two-year old machines are the same under both Omicron and Chi. The rent of type-2 land, which is non-scarce under Omicron, is zero under Chi, as it would be at a non-fluke switch point.

**Figure 7: Demand for Labor for an Example with Fixed Capital and Rent**




The presentation of the variation of employment with the wage, given final demand (Figure 7) concludes the exposition of this numerical example. It is an example of capital reversing. The wage curves for Alpha and Beta, that is, for Pi and Rho, intersect twice. Only the intersection at the larger rate of profits is on the wage frontier. Around this switch point, a higher wage is associated with firms adopting a technique that requires greater employment throughout the economy.

## 6. Conclusion

This article presents a number of worked out examples for prices of production. Although not a focus of this article, these examples demonstrate that capital reversing is compatible with the existence of extensive and intensive rent, the production of multiple agricultural commodities, and the use of fixed capital on scarce land. I find it difficult to understand the continual insistence by some economists on simple supply and demand stories in, for example, long run models of wages and employment.

The numerical examples illustrate that switch points can lie along a single wage curve, with no intersecting wage curve and with variation in which processes are operated between the cost-minimizing techniques. Switch points can exist in which wage curves intersect, and no variation occurs in which processes are operated between the cost-minimizing techniques. Switch points like these occur in the examples with the intersections of two and three wage curves. The rate of profits at which fake switch points occur can be independent of the choice of numeraire and dependent on the numeraire in other cases. Intersections of wage curves can be fake switch points because, although the prices of all commodities produced under both techniques do not vary between the techniques, the prices of ghost commodities under one technique are positive at the fake.

The concept of a switch point is not tightly tied to intersections of wage curves in models of joint production. The examples demonstrate that some properties of single production, specifically related to switch points, can be relaxed in joint production without losing decreasing wage frontiers or the property of the cost-minimizing technique being unique and square away from switch points. Perhaps future research can clarify in which models of joint production these properties fail. A sufficient condition for the existence of a cost-minimizing technique is that a given consumption vector can be feasibly produced at a rate of growth equal to the given rate of profits (Lemma 8.2 in chapter 8 of Kurz and Salvadori 1995: 230). I think more can be said about models of extensive and intensive rent. I speculate that a sufficient condition for a decreasing wage frontier and a unique, square cost-minimizing technique is that the equations for the solving subsystem for each cost-minimizing technique embody a Sraffa matrix (Kurz and Salvadori 1995: 123-124).

## Appendix A: Further Details for the Examples

This appendix provides selected justification for claims in the main text. A more complete and even more tedious presentation would provide explicit solutions for quantity and price equations for all techniques in all examples.

### *A.1 Reswitching Example with Extensive Rent*

Consider the reswitching example in Section 3. Required net output is 125 bushels of corn. This appendix demonstrates that this level of final demand requires farming both types of land to some extent. Endowments consist of 100 acres for each.

Under Alpha, corn is grown on type-1 land, up to the limit of its endowment. Under Theta, corn is grown on type-1 land at the limit of its endowment, and additional corn, if needed for final demand, is grown on type-2 land. Table 1 shows the quantity flows, at the boundary for these two techniques, when net output consists solely of corn. Input entries of zero are not shown. The output of iron from the first process replaces the iron inputs. The net output is $890/11 \approx 80.9$ bushels of corn, produced by a vertically integrated industry from an input of $1{,}040/11 \approx 94.5$ person-years.

**Table A-1: Quantity Flows for Extremes of Alpha and Theta**

| Inputs | I | II | III |
|---|---|---|---|
| Labor (Person-Yrs.) | $a_{0,1} \cdot q_1 = 50/11$ | $a_{0,2} \cdot q_2 = 90$ | |
| Type-1 Land (Acres) | | $c_{1,2} \cdot q_2 = 100$ | |
| Type-2 Land (Acres) | | | |
| Iron (Tons) | $a_{1,1} \cdot q_1 = 45/22$ | $a_{1,2} \cdot q_2 = 5/2$ | |
| Corn (Bushels) | $a_{2,1} \cdot q_1 = 100/11$ | $a_{2,2} \cdot a_2 = 10$ | |
| OUTPUTS | $q_1 = 50/11$ | $q_2 = 100$ | $q_3 = 0$ |

Table 2 shows the corresponding quantity flows for Epsilon to produce the same net output. Here type-2 land is farmed to the extent of its endowment. Labor input summed over all three processes is $4{,}370{,}990/47{,}971 \approx 91.1$ person-years. For final demand to consist solely of corn and to require farming both types of land, whatever feasible technique is adopted, it must consist of more than approximately 80.9 bushels of corn. Theta requires more labor input than Epsilon over the range of net output with this lower bound.

**Table A-2: Quantity Flows for Epsilon for Same Net Output**

| Inputs | I | II | III |
|---|---|---|---|
| Labor (Person-Yrs.) | $a_{0,1} \cdot q_1 = \frac{81{,}650}{47{,}971}$ | $a_{0,2} \cdot q_2 = \frac{122{,}940}{4{,}361}$ | $a_{0,3} \cdot q_3 = \frac{3{,}000}{49}$ |
| Type-1 Land (Acres) | | $c_{1,2} \cdot q_2 = \frac{136{,}600}{4{,}361}$ | |
| Type-2 Land (Acres) | | | $c_{2,3} \cdot q_3 = 100$ |
| Iron (Tons) | $a_{1,1} \cdot q_1 = \frac{73{,}485}{95{,}942}$ | $a_{1,2} \cdot q_2 = \frac{3{,}415}{4{,}361}$ | $a_{1,3} \cdot q_3 = \frac{15}{98}$ |
| Corn (Bushels) | $a_{2,1} \cdot q_1 = \frac{163{,}300}{47{,}971}$ | $a_{2,2} \cdot a_2 = \frac{13{,}660}{4{,}361}$ | $a_{2,3} \cdot a_3 = \frac{2{,}250}{49}$ |
| OUTPUTS | $q_1 = \frac{81{,}650}{47{,}971}$ | $q_2 = \frac{136{,}600}{4{,}361}$ | $q_3 = \frac{5{,}000}{49}$ |

Table A-3 shows the upper bound for quantity flows for Epsilon and Theta. Final demand consists solely of corn. Both types of land are fully farmed. Net output is $73{,}560/539 \approx 136.5$ bushels of corn. This is the upper bound on how much corn can be produced for a self-reproducing economy in this example.

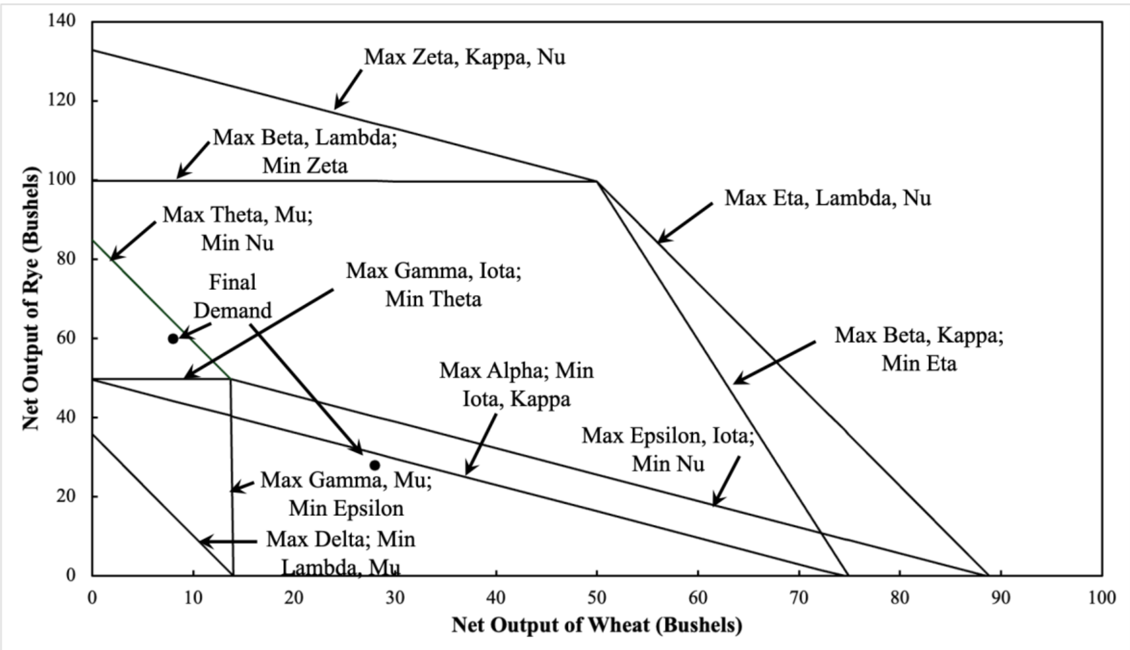
**Table A-3: Quantity Flows for Epsilon and Theta Maximum for Net Output**

| Inputs | I | II | III |
|---|---|---|---|
| Labor (Person-Yrs.) | $a_{0,1} \cdot q_1 = \dfrac{2{,}600}{539}$ | $a_{0,2} \cdot q_2 = 90$ | $a_{0,3} \cdot q_3 = \dfrac{3{,}000}{49}$ |
| Type-1 Land (Acres) | | $c_{1,2} \cdot q_2 = 100$ | |
| Type-2 Land (Acres) | | | $c_{2,3} \cdot q_3 = 100$ |
| Iron (Tons) | $a_{1,1} \cdot q_1 = \dfrac{1{,}170}{539}$ | $a_{1,2} \cdot q_2 = \dfrac{5}{2}$ | $a_{1,3} \cdot q_3 = \dfrac{15}{98}$ |
| Corn (Bushels) | $a_{2,1} \cdot q_1 = \dfrac{5{,}200}{539}$ | $a_{2,2} \cdot a_2 = 10$ | $a_{2,3} \cdot a_3 = \dfrac{2{,}250}{49}$ |
| OUTPUTS | $q_1 = \dfrac{2{,}600}{539}$ | $q_2 = 100$ | $q_3 = \dfrac{5{,}000}{49}$ |

### *A.2 Reswitching Example with Extensive and Intensive Rent*

Two final demands are initially considered for the example in Section 4, in which multiple agricultural commodities are produced. In both cases, the final demand consists of no iron and specified quantities of wheat and rye. Figure A-1 partitions the space for final demands based on the minimum and maximum feasible final demand for wheat and rye for each of the 13 available techniques. The two cases which are analyzed for final demand are also shown.

**Figure A-1: Feasible Techniques for Given Final Demands**



*A.2.1 Example with One Specification of Final Demand*

Suppose final demand consists of 8 bushels of wheat and 60 bushels of rye, as in Section 4.1. Figure 6 shows that the Beta, Theta, Kappa, Lambda, and Mu techniques are feasible.

The rate of profits, the wage, prices, and rents for a technique can be used to calculate the costs of the inputs of processes not operated under the technique. The difference between revenues and costs, when operated at a unit level, is extra profits. Costs include a charge for the given rate of profits on the value of the capital goods advanced. A technique is cost-minimizing only if no process pays extra profits. Figure A-2 illustrates for the Beta technique. Processes I, II, and V are operated under the Beta technique. Extra profits, at Beta prices, are plotted in the figure for the other two processes in the technology. Beta is never cost-minimizing, whatever the rate of profits.

**Figure A-2: Extra Profits for the Beta Technique**

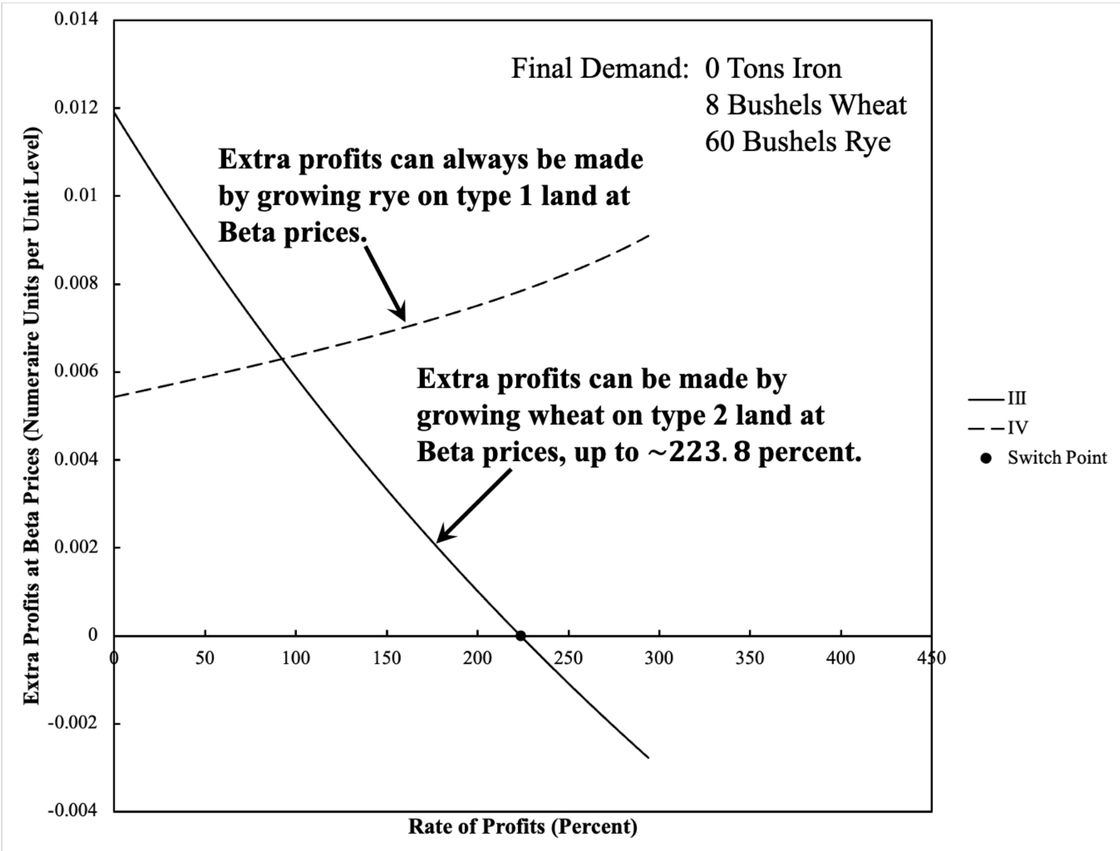
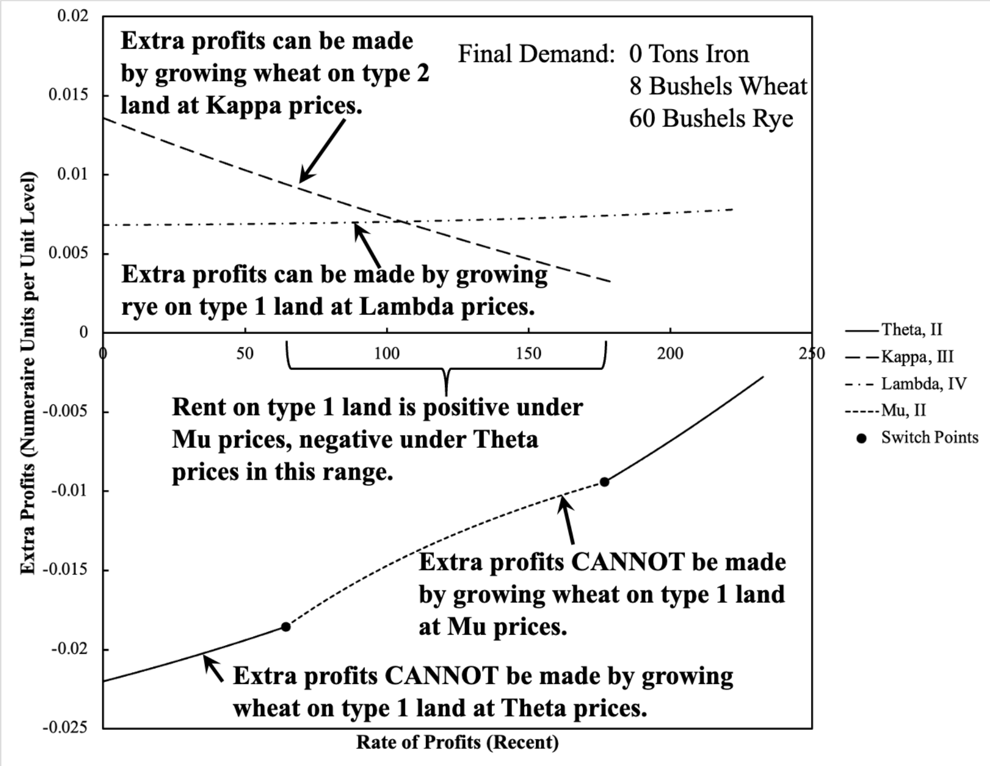



Figure A-3 shows extra profits for non-operated processes for the remaining techniques that are feasible at the given final demand. Each plotted function is drawn for the price system for the indicated technique. The graph demonstrates that only Theta and Mu are cost-minimizing, in non-overlapping ranges of the rate of profits.

**Figure A-3: Extra Profits for Other Feasible Techniques**




*A.2.2 Example with Another Specification of Final Demand*

Section 4.2 provides an exposition of this example with an initial final demand of 28 bushels of wheat and 28 bushels of rye. Alpha, Beta, Epsilon, and Lambda are feasible. A switch point between Alpha and Epsilon is considered. The wage curves for Gamma and Iota also intercept with the wage curve for Alpha at the switch point. The wage curve for Alpha is also the wage curve for Epsilon and Zeta, since they all have the same solving subsystem.

Figure A-4 shows the extra profits available under Alpha for the wheat-growing and rye-growing process not operated under the technique. For a rate of profits between zero and the switch point under consideration, extra profits are obtained at Alpha prices by growing wheat on type-2 land. For rates of profits between the switch point and the maximum, Alpha is cost-minimizing and both types of land are free. Figures A-2 and A-3 above show that Beta and Lambda, respectively, are never cost-minimizing. Figure A-5 plots extra profits available under selected techniques from operating processes not part of the techniques. Epsilon is cost-minimizing for the range of the rate of profits in which landlords are paid rent on type-2 land under Epsilon prices.

Section 4.2 also considers the switch point between Epsilon and Alpha at a lower and slightly higher level of final demand. At a low enough level, Alpha, Beta, Gamma, and Delta are feasible. As illustrated in Figure A-5, Gamma would then be cost-minimizing for rates of profits slightly below that at the switch point. Alpha would remain cost-minimizing for rates of profits above the switch point.

**Figure A-4: Extra Profits for the Alpha Technique**

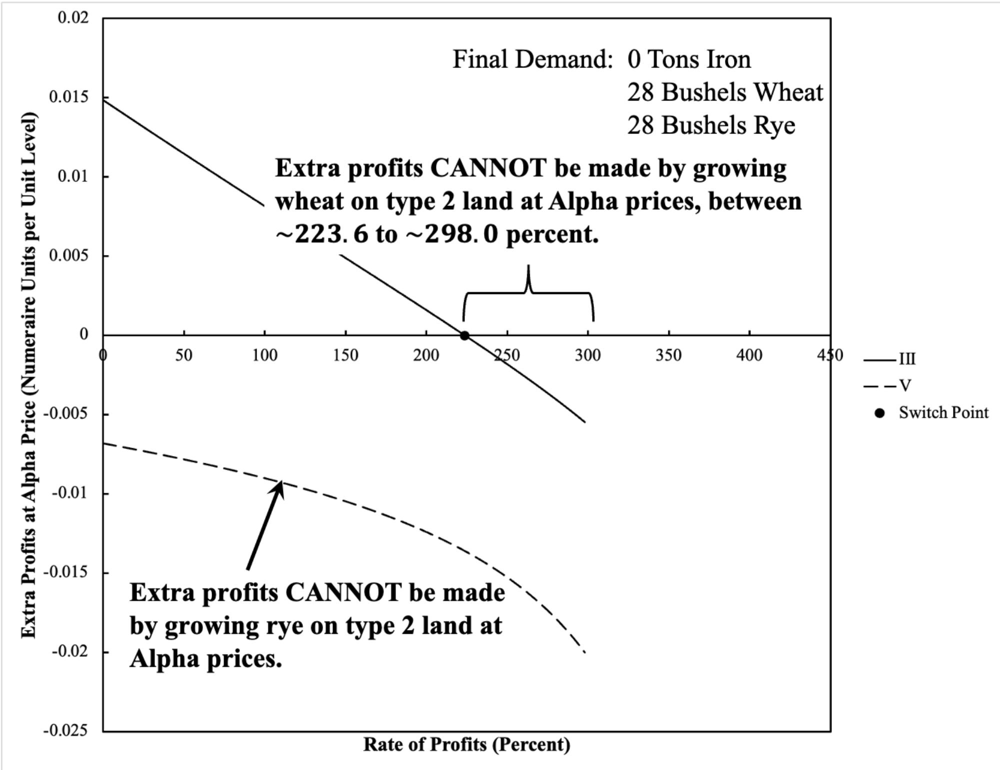
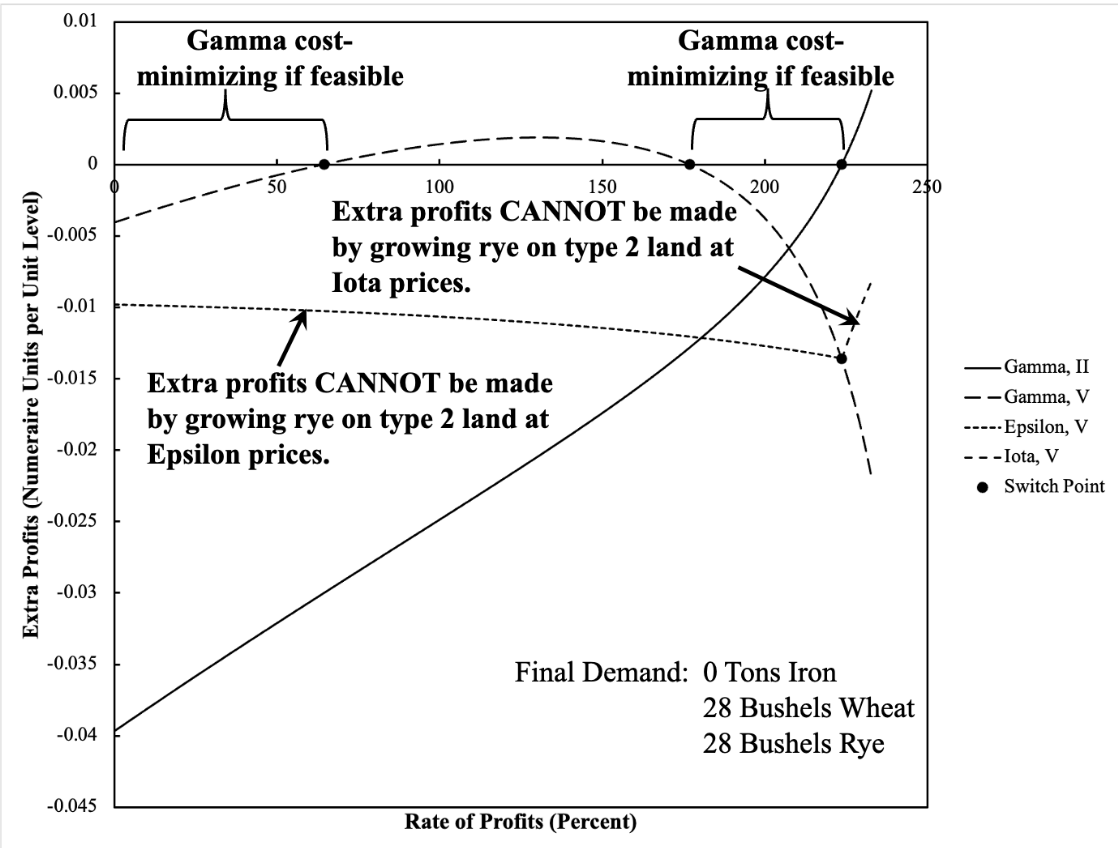


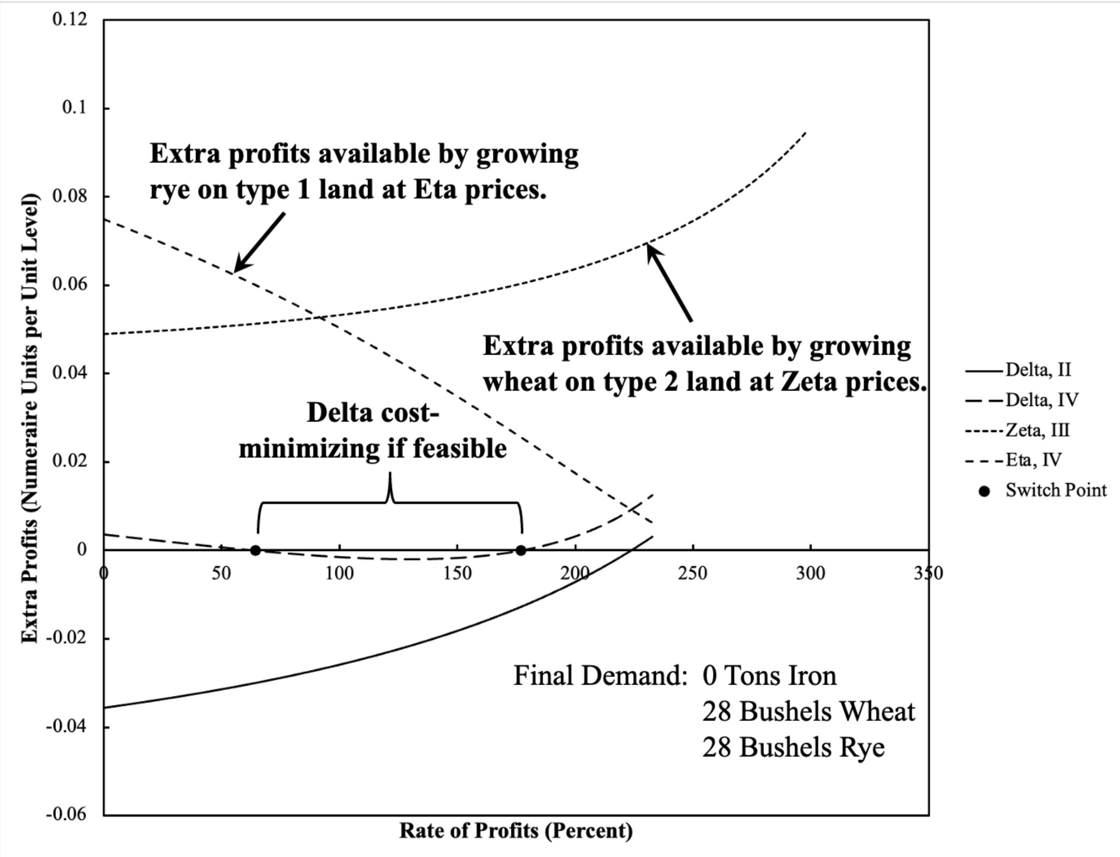

**Figure A-5: Extra Profits for Other Techniques with Wage Curves at the Switch Point**



Beta, Epsilon, Iota, Kappa, and Lambda are feasible when final demand consists of 30 bushels of wheat and 30 bushels of rye. Epsilon is cost-minimizing at rates of profit lower than at the switch point. Figure A-5 shows that Iota is cost-minimizing at rates of profits above the switch point.

A fake switch point, at a rate of profits of approximately 219 percent, is considered. The rate of profits at which the wage curves for Aloha and Delta intersect varies with the numeraire. Alpha, Epsilon, and Zeta have the same solving subsystem, as do Delta, Eta, and Theta. Figure A-6 graphs extra profits for the techniques with wage curves for the fake that have not yet been graphed. Only Epsilon is cost-minimizing around the fake.

**Figure A-6: Extra Profits for Selected Techniques with Wage Curves at Fake Switch Point**




*A.3 Capital-Reversing Example with Fixed Capital and Rent*

The analysis of the choice of technique in models of fixed capital requires identifying when the price of old machines is positive. Accordingly, the section of the appendix plots prices for old machines as functions of the rate of profits for the example in Section 5. Figures A-7 and A-8 plot the price of one-year old and two-year old machines, respectively, by technique.

**Figure A-7: Prices of Type-1 One-Year Old Machines**

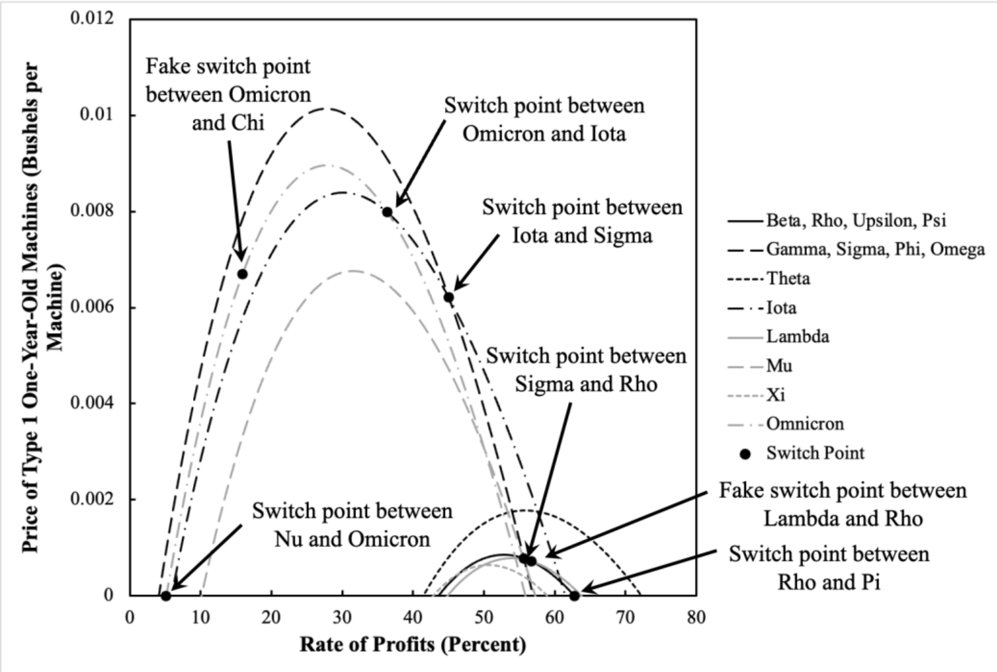
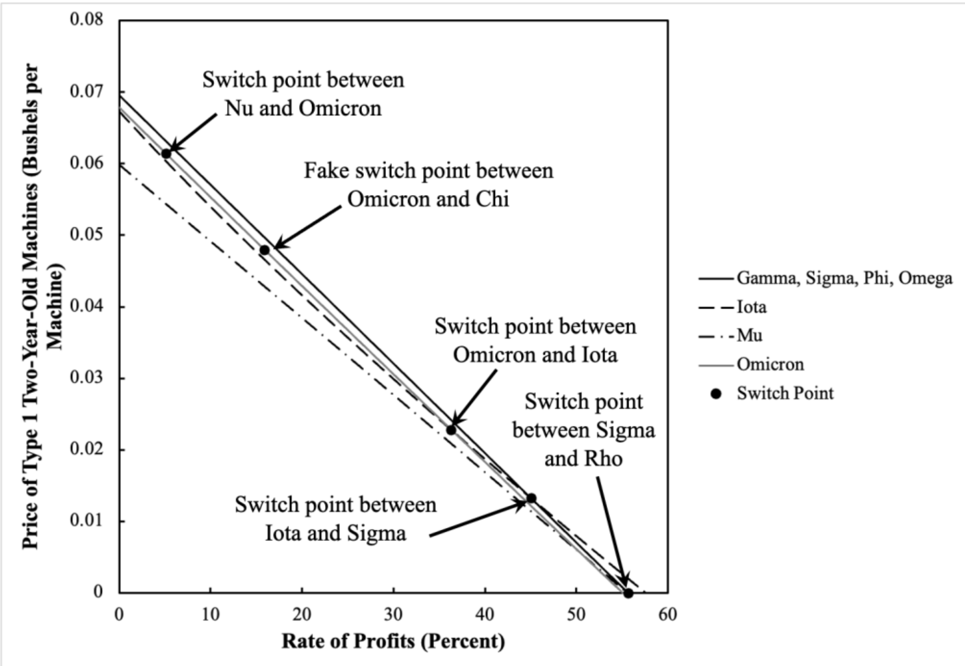



**Figure A-8: Prices of Type-1 Two-Year Old Machines**




Figures A-9 and A-10 plot prices of one-year old and two-year old type-2 machines. Prices, as a function of the rate of profits, are only graphed for techniques in which the machine is operated and the price is positive for some range of the rate of profits. Figures A-11 and A-12 plot rent curves for the example.

**Figure A-9: Prices of Type-2 One-Year Old Machines**

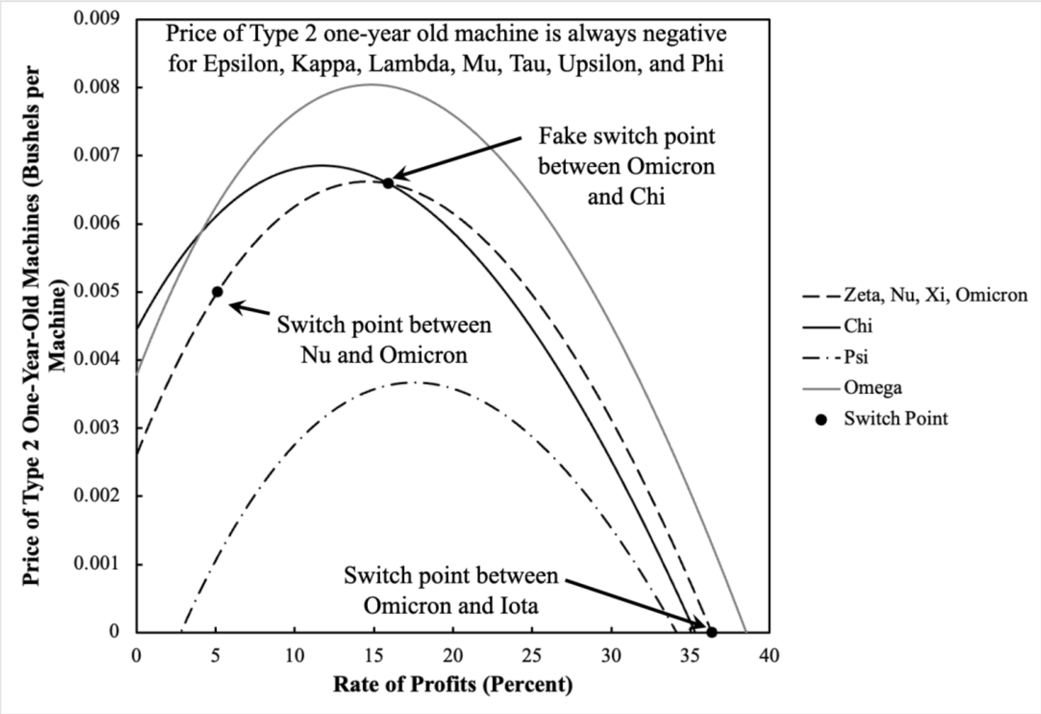



**Figure A-10: Prices of Type-2 Two-Year Old Machines**

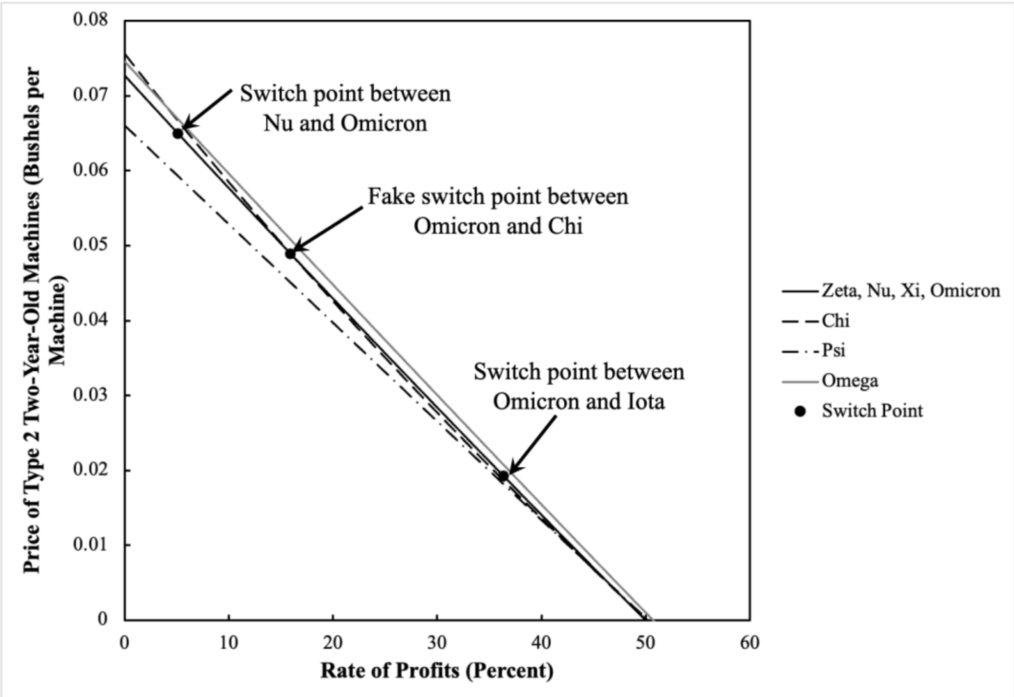


**Figure A-11: Rent on Type-1 Land**

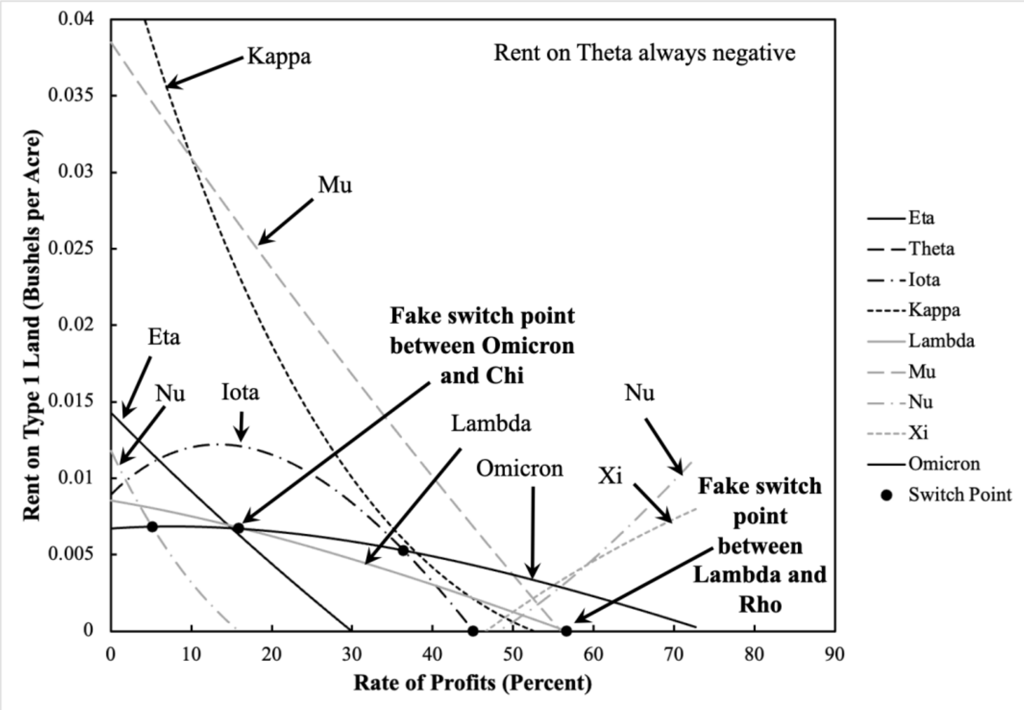



**Figure A-12: Rent on Type-2 Land**

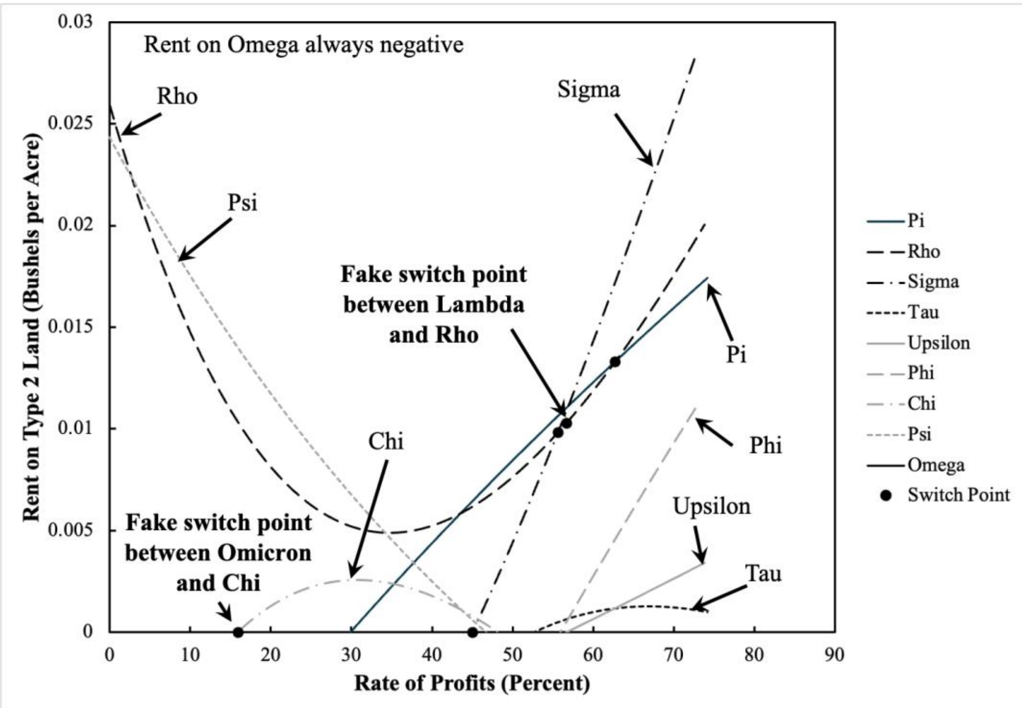



Table A-4 shows for techniques in which type-1 land is scarce, the ranges of the rate of profits in which the prices of produced machines are positive and the range in which rent is positive. Table A-5 shows these ranges for the rate of profits for techniques in which type-2 land is scarce. The endpoints of these ranges are reported as rounded. The range of the rate of profits, for a technique, in which the wage is positive is identical to the range of the rate of profits in which the price of the new machine is positive. These tables can be used to find the cost-minimizing technique for each rate of profits, when final demand is such that one type of land is scarce.

**Table A-4: Choice of Technique for Example with Fixed Capital, Type-1 Land Scarce**

| Technique | Processes | Range (Percent) | Condition |
|---|---|---|---|
| Eta | I, II, V | $0 \leq r \leq 76.5$ | Price of new machine is positive. |
| | | $0 \leq r < 29.9$ | Rent on type-1 land is positive. |
| Theta | I, II, III, V | N/A | Rent on type-1 land is always negative. |
| Iota | I, II, III, IV, V | $0 \leq r \leq 76.5$ | Price of new machine is positive. |
| | | $5.9 < r < 61.5$ | Price of type-1 one-yr. old machine is positive. |
| | | $0 \leq r < 57.6$ | Price of type-1 two-yr. old machine is positive. |
| | | $0 \leq r < 45.0$ | Rent on type-1 land is positive. |
| Kappa | I, II, V, VI | N/A | Price of type-2 one-yr. old machine is always negative. |
| Lambda | I, II, III, V, VI | N/A | Price of type-2 one-yr. old machine is always negative. |
| Mu | I, II, III, IV, V, VI | N/A | Price of type-2 one-yr. old machine is always negative. |
| Nu | I, II, V, VI, VII | $0 \leq r \leq 72.6$ | Price of new machine is positive. |
| | | $0 \leq r < 36.3$ | Price of type-2 one-yr. old machine is positive. |
| | | $0 \leq r < 50.0$ | Price of type-2 two-yr. old machine is positive. |
| | | $0 \leq r < 15.9$ or $48.2 < r \leq 72.6$ | Rent on type-1 land is positive. |
| Xi | I, II, III, V, VI, VII | $0 \leq r \leq 72.6$ | Price of new machine is positive. |
| | | $42.8 < r < 58.9$ | Price of type-1 one-yr. old machine is positive. |
| | | $0 \leq r < 36.3$ | Price of type-2 one-yr. old machine is positive. |
| | | $0 \leq r < 50.0$ | Price of type-2 two-yr. old machine is positive. |
| | | $46.6 < r \leq 72.6$ | Rent on type-1 land is positive. |
| Omicron | I, II, III, IV, V, VI, VII | $0 \leq r \leq 72.6$ | Price of new machine is positive. |
| | | $5.1 < r < 55.8$ | Price of type-1 one-yr. old machine is positive. |
| | | $0 \leq r < 55.1$ | Price of type-1 two-yr. old machine is positive. |
| | | $0 \leq r < 36.3$ | Price of type-2 one-yr. old machine is positive. |
| | | $0 \leq r < 50.0$ | Price of type-2 two-yr. old machine is positive. |
| | | $0 \leq r < 72.6$ | Rent on type-1 land is positive. |

**Table A-5: Choice of Technique for Example with Fixed Capital, Type-2 Land Scarce**

| Technique | Processes | Range (Percent) | Condition |
|---|---|---|---|
| Pi | I, II, V | $0 \le r \le 74.2$ | Price of new machine is positive. |
| | | $29.9 < r \le 74.2$ | Rent on type-2 land is positive. |
| Rho | I, II, III, V | $0 \le r \le 73.8$ | Price of new machine is positive. |
| | | $43.6 < r < 62.7$ | Price of type-1 one-yr. old machine is positive. |
| | | $0 \le r \le 73.8$ | Rent on type-2 land is positive. |
| Sigma | I, II, III, IV, V | $0 \le r \le 72.7$ | Price of new machine is positive. |
| | | $4.1 < r < 56.9$ | Price of type-1 one-yr. old machine is positive. |
| | | $0 \le r < 55.7$ | Price of type-1 two-yr. old machine is positive. |
| | | $45.0 < r \le 72.7$ | Rent on type-2 land is positive. |
| Tau | I, II, V, VI | N/A | Price of type-2 one-yr. old machine is always negative. |
| Upsilon | I, II, III, V, VI | N/A | Price of type-2 one-yr. old machine is always negative. |
| Phi | I, II, III, IV, V, VI | N/A | Price of type-2 one-yr. old machine is always negative. |
| Chi | I, II, V, VI, VII | $0 \le r \le 74.2$ | Price of new machine is positive. |
| | | $0 \le r < 35.3$ | Price of type-2 one-yr. old machine is positive. |
| | | $0 \le r < 50.1$ | Price of type-2 two-yr. old machine is positive. |
| | | $15.9 < r < 48.2$ | Rent on type-2 land is positive. |
| Psi | I, II, III, V, VI, VII | $0 \le r \le 73.8$ | Price of new machine is positive. |
| | | $43.6 < r < 62.7$ | Price of type-1 one-yr. old machine is positive. |
| | | $2.8 < r < 34.1$ | Price of type-2 one-yr. old machine is positive. |
| | | $0 \le r < 50.2$ | Price of type-2 two-yr. old machine is positive. |
| | | $0 \le r < 46.6$ | Rent on type-2 land is positive. |
| Omega | I, II, III, IV, V, VI, VII | N/A | Rent on type-2 land is always negative |

A candidate range of profits for which a technique might be cost minimizing (Table A-6) is the intersection of the range of the rate of profits in the table with an entry for that technique. That intersection is empty for Xi and Psi. These techniques are never cost-minimizing. For a given technique, identify those techniques in which the economic life of either type of machine is extended. The given technique is cost-minimizing at a rate of profits if and only if that rate of profits is not in the candidate range for these other techniques.

**Table A-6: Candidate Approximate Ranges of the Rate of Profits by Technique**

| Type-1 Land | | Type-2 Land | |
|---|---|---|---|
| **Technique** | **Range (Percent)** | **Technique** | **Range (Percent)** |
| Eta | $0 \leq r < 29.9$ | Pi | $29.9 < r < 74.2$ |
| Theta | N/A | Rho | $43.6 < r < 62.7$ |
| Iota | $5.9 < r < 45.0$ | Sigma | $45.0 < r < 55.7$ |
| Kappa | N/A | Tau | N/A |
| Lambda | N/A | Upsilon | N/A |
| Mu | N/A | Phi | N/A |
| Nu | $0 \leq r < 15.9$ | Chi | $15.9 < r < 35.3$ |
| Xi | N/A | Psi | N/A |
| Omicron | $5.1 < r < 36.3$ | Omega | N/A |

For example, consider Pi, in which both types of machines are run for one year. Type-1 machines are otherwise operated for two years under Theta and Rho. From Rho, the candidate range for Pi is reduced to $29.9 < r < 43.6$ or $62.7 < r < 74.2$ percent. Iota and Sigma operate type-1 machines for three years. Iota, in which type-1 land is scarce, dominates Pi in the first interval. Type-2 machines are extended to two years, as compared to Pi, in Kappa and Iota. They are extended to three years under Nu and Chi. None of these techniques further limit when Pi is cost-minimizing. The cost-minimizing techniques are, in order of an increasing rate of profits, Nu, Omicron, Iota, Sigma, Rho, and Pi.

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